MATH 315 Fall 2024

Assignment 11

Due: Friday, October 18

Part I: The original Lotka-Volterra Predator Prey Model

$$\frac{dx}{dt} = ax - bxy,$$
$$\frac{dy}{dt} = mxy - ny$$

has four parameters a, b, m ,n.

It is possible to make simple invertible linear transformations of the variables to create an equivalent system with the same dynamics but fewer parameters.

Toward this end, let

$$X = \frac{m}{n}x, \qquad Y = \frac{b}{a}y, \qquad \text{and} \quad T = at$$

1). Show that these change of variables converts the original system to

$$\frac{dX}{dT} = X - XY, \qquad \frac{dY}{dT} = \gamma(XY - Y)$$

where $\gamma = \frac{n}{a}$.

Hint: It may be useful to use the Chain Rule to write

$$\frac{dX}{dT} = \frac{dX}{dx}\frac{dx}{dt}\frac{dt}{dT}$$

2) Use our qualitative and quantitative techniques to analyze this simpler system. [Note for instance that the stable points are (0,0) and (1,1)]. Replicate as much as possible the approach we used on the original Lotka – Volterra model, including finding and solving the linear system that approximates the behavior near the stable point (1,1). How are the trajectories of this system affected by different values of γ?

Part II: Write up solutions for Problems 11 and 12 on the attached excerpt from the Brannon and Boyce differential equations text.

Bifurcation Points. For an equation of the form

$$dy/dt = f(a,y) \qquad \text{(i)}$$

where a is a real parameter, the critical points (equilibrium solutions) usually depend on the value of a. As a steadily increases or decreases, it often happens that at a certain value of a, called a **bifurcation point**, critical points come together, or separate, and equilibrium solutions may either be lost or gained.
Bifurcation points are of great interest in many applications, because near them the nature of the solution of the underlying differential equation is undergoing an abrupt change. For example, in fluid mechanics a smooth (laminar) flow may break up and become turbulent. Or an axially loaded column may suddenly buckle and exhibit a large lateral displacement. Or, as the amount of one of the chemicals in a certain mixture is increased, spiral wave patterns of varying color may suddenly emerge in an originally quiescent fluid.

Problems 10 through 12 describe three types of bifurcations that can occur in simple equations of the form (i).

10. Consider the equation

$$dy/dt = a - y^2 \qquad \text{(ii)}$$

(a) Find all of the critical points for Eq. (ii). Observe that there are no critical points if $a < 0$, one critical point if $a = 0$, and two critical points if $a > 0$.
(b) Draw the phase line in each case and determine whether each critical point is asymptotically stable, semistable, or unstable.
(c) In each case, sketch several solutions of Eq. (ii) in the ty-plane.
(d) If we plot the location of the critical points as a function of a in the ay-plane, we obtain Figure 2.5.11. This is called the **bifurcation diagram** for Eq. (ii). The bifurcation at $a = 0$ is called a **saddle–node bifurcation**. This name is more natural in the context of second order systems, which are discussed in Chapter 7.

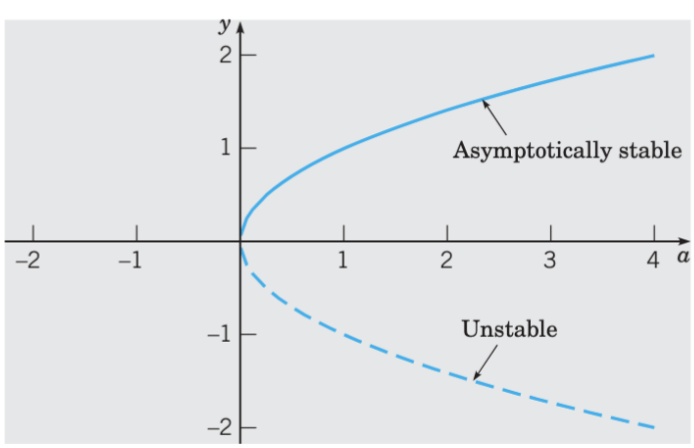

FIGURE 2.5.11 Bifurcation diagram for $y' = a - y^2$.

11. Consider the equation

$$\frac{dy}{dt} = ay - y^3 = ay - y^3 = y(a - y^2) \qquad \text{(iii)}$$

(a) Again consider the cases $a < 0$, $a = 0$, and $a > 0$. In each case, find the critical points, draw the phase line, and determine whether each critical point is asymptotically stable, semistable, or unstable.

(b) In each case, sketch several solutions of Eq. (iii) in the *ty*-plane.

(c) Draw the bifurcation diagram for Eq. (iii), that is, plot the location of the critical points versus a. For Eq. (iii), the bifurcation point at $a = 0$ is called a **pitchfork bifurcation**; your diagram may suggest why this name is appropriate.

12. Consider the equation

$$\frac{dy}{dy} = ay - y^2 = y(a - y) \qquad \text{(iv)}$$

(a) Again consider the cases $a < 0$, $a = 0$, and $a > 0$. In each case, find the critical points, draw the phase line, and determine whether each critical point is asymptotically stable, semistable, or unstable.

(b) In each case, sketch several solutions of Eq. (iv) in the *ty*-plane.

(c) Draw the bifurcation diagram for Eq. (iv). Observe that for Eq. (iv) there are the same number of critical points for $a < 0$ and $a > 0$ but that their stability has changed. For $a < 0$, the equilibrium solution $y = 0$ is asymptotically stable and $y = a$ is unstable, while for $a > 0$ the situation is reversed. Thus there has been an **exchange of stability** as a passes through the bifurcation point $a = 0$. This type of bifurcation is called a **transcritical bifurcation**.

This excerpt is taken from the Exercises at the end of Section 2.5 ("Autonomous Equations and Population Dynamics") on pages 92 and 93 of *Differential Equations: An Introduction to Modern Methods and Applications,* Third Edition,, by James Brannan and William Boyce. There are copies of this text on two hour reserve at the Davis Family Library. You can also find the entire book as a pdf at https://files.owenoertell.com/textbooks/math/differential-brannan.pdf