Bifurcation Points. Consider the system

$$x' = F(x, y, \alpha), \qquad y' = G(x, y, \alpha), \tag{i}$$

where α is a parameter. The equations

$$F(x, y, \alpha) = 0, \qquad G(x, y, \alpha) = 0 \tag{ii}$$

determine the x and y nullclines, respectively. Any point where an x nullcline and a y nullcline intersect is a critical point. As α varies and the configuration of the nullclines changes, it may well happen that, at a certain value of α, two critical points coalesce into one. For further variations in α, the critical point may disappear altogether, or two critical points may reappear, often with stability characteristics different from before they coalesced. Of course, the process may occur in the reverse order. For a certain value of α, two formerly nonintersecting nullclines may come together, creating a critical point, which, for further changes in α, may split into two. A value of α at which critical points coalesce, and possibly are lost or gained, is a bifurcation point. Since a phase portrait of a system is very dependent on the location and nature of the critical points, an understanding of bifurcations is essential to an understanding of the global behavior of the system's solutions. Problems 11 through 17 illustrate some possibilities that involve bifurcations.

In each of Problems 11 through 14:

(a) Sketch the nullclines and describe how the critical points move as α increases.

(b) Find the critical points.

(c) Let $\alpha = 2$. Classify each critical point by investigating the corresponding approximate linear system. Draw a phase portrait in a rectangle containing the critical points.

(d) Find the bifurcation point α_0 at which the critical points coincide. Locate this critical point and find the eigenvalues of the approximate linear system. Draw a phase portrait.

(e) For $\alpha > \alpha_0$, there are no critical points. Choose such a value of α and draw a phase portrait.

11. $x' = -6x + y + x^2, \qquad y' = \frac{3}{2}\alpha - y$

12. $x' = \frac{3}{2}\alpha - y, \qquad y' = -x + y + x^2$

13. $x' = -3x + y + x^2, \qquad y' = -\alpha - x + y$

14. $x' = -\alpha - 2x + y, \qquad y' = -4x + y + x^2$

In each of Problems 15 and 16:

(a) Find the critical points.

(b) Determine the value of α, denoted by α_0, where two critical points coincide.

(c) By finding the approximating linear systems and their eigenvalues, determine how the stability properties of these two critical points change as α passes through the bifurcation point α_0.

(d) Draw phase portraits for values of α near α_0 to show how the transition through the bifurcation point occurs.

15. $x' = (3 + x)(1 - x + y),$
$y' = (y - 1)(1 + x + \alpha y); \qquad \alpha > 0$

16. $x' = y(\alpha - 2x + 3y), \qquad y' = (4 - x)(3 + y)$

17. Suppose that a certain pair of competing species are described by the system

$$\frac{dx}{dt} = x(4 - x - y),$$
$$\frac{dy}{dt} = y(2 + 2\alpha - y - \alpha x),$$

where $\alpha > 0$ is a parameter.

(a) Find the critical points. Note that (2, 2) is a critical point for all values of α.

(b) Determine the nature of the critical point (2, 2) for $\alpha = 0.75$ and for $\alpha = 1.25$. There is a value of α between 0.75 and 1.25 where the nature of the critical point changes abruptly. Denote this value by α_0; it is also called a bifurcation point.

(c) Find the approximate linear system near the point (2, 2) in terms of α.

(d) Find the eigenvalues of the linear system in part (c) as functions of α. Then determine α_0.

(e) Draw phase portraits near (2, 2) for $\alpha = \alpha_0$ and for values of α slightly less than, and slightly greater than, α_0. Explain how the transition in the phase portrait takes place as α passes through α_0.