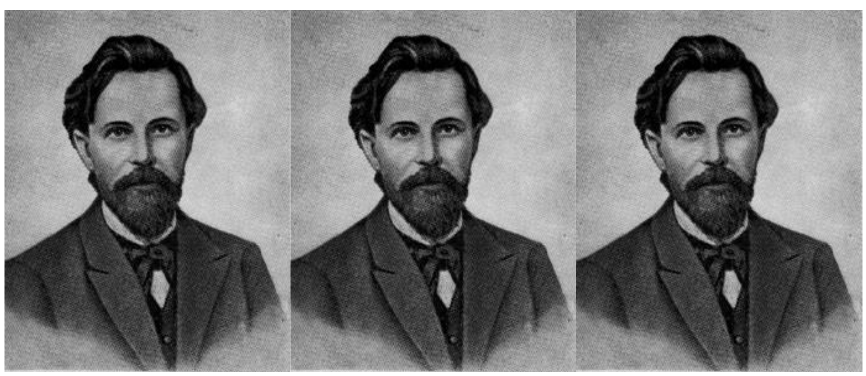
Stochastic Processes and Markov Chains III



Absorbing Markov Chains

Example 1: Consider flipping a coin with probability p for Heads.

Steps: Successive Flips

States: **A** = { Flipped At Least 1 Head }
T = { No Heads} (All Tails)

Observation: Once you enter state **A**, you never leave.

This process is a Markov chain with transition matrix P where

$$P = \begin{array}{c} \\ \text{A} \\ \text{T} \end{array} \begin{array}{c} \begin{array}{cc} \text{A} & \text{T} \end{array} \\ \begin{pmatrix} 1 & 0 \\ p & q \end{pmatrix} \end{array} \quad q = 1 - p$$

$$P = \begin{array}{c} \\ A \\ T \end{array} \begin{array}{c} \begin{array}{cc} A & T \end{array} \\ \begin{pmatrix} 1 & 0 \\ p & q \end{pmatrix} \end{array} \quad q = 1 - p$$

Then $P^2 = \begin{pmatrix} 1 & 0 \\ p & q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p & q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ p+qp & q^2 \end{pmatrix} =$

$$\begin{pmatrix} 1 & 0 \\ 1-q^2 & q^2 \end{pmatrix}$$

since $p + qp = (1 - q) + q(1-q)$

$$= 1 - q + q - q^2 = 1 - q^2$$

Then $P^3 = \begin{pmatrix} 1 & 0 \\ p & q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p+qp & q^2 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ p + qp + q^2 p & q^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1-q^3 & q^3 \end{pmatrix}$$

In general, $P^n = \begin{pmatrix} 1 & 0 \\ 1\text{-}q^n & q^n \end{pmatrix}$ =

$$\begin{pmatrix} 1 & 0 \\ p + qp + q^2p + \cdots + q^{n-1} & q^n \end{pmatrix}$$

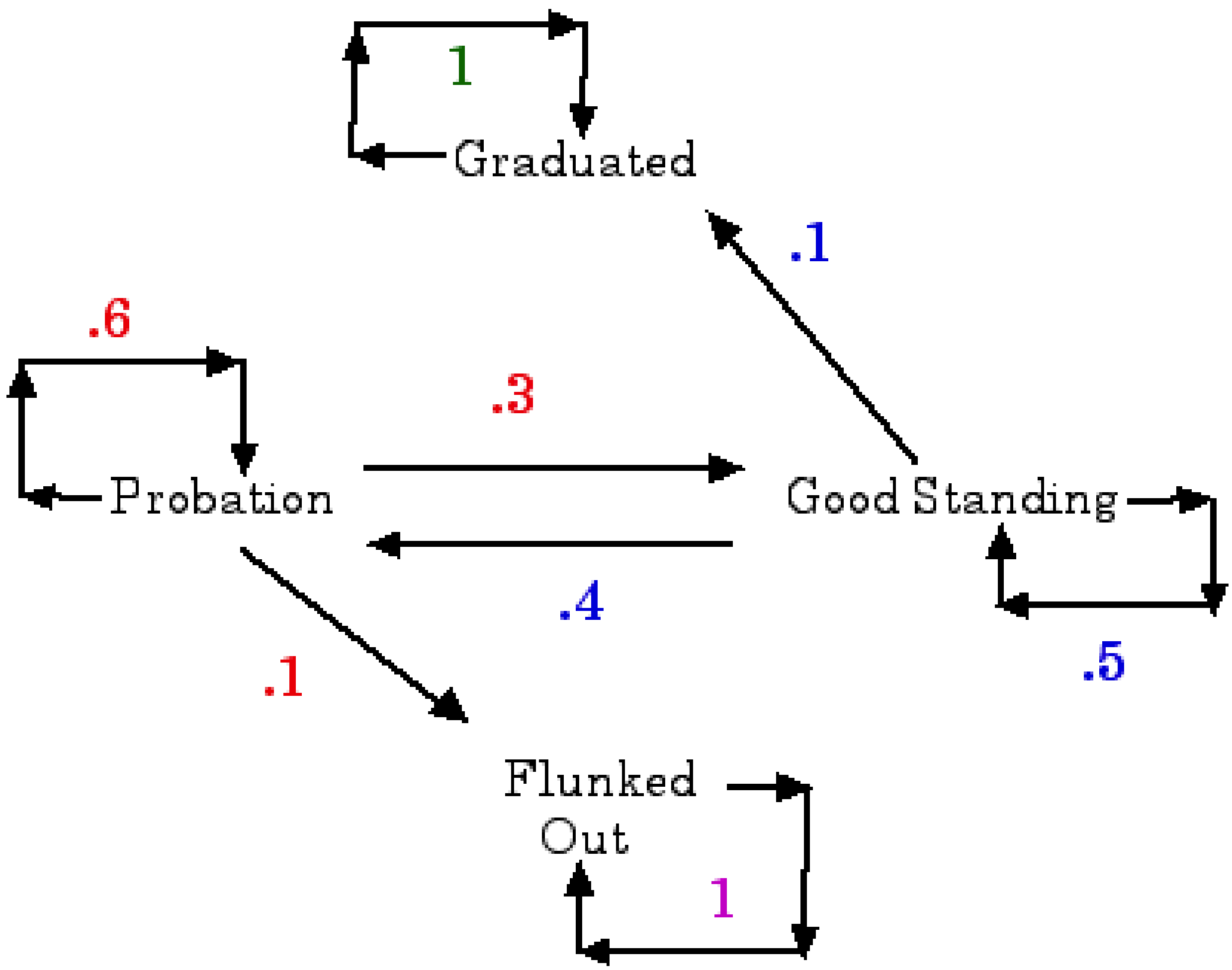
In the limit, as $n \rightarrow \infty$,

$p+qp+q^2p+\ldots+q^{n\text{-}1}p$

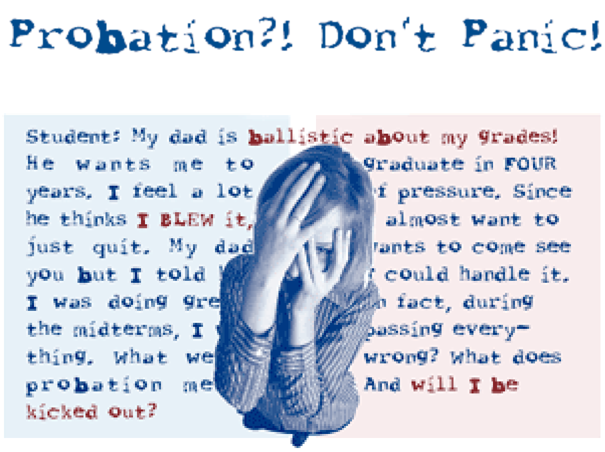
$= p(1+q+q^2+\ldots+q^{n\text{-}1}+..)$

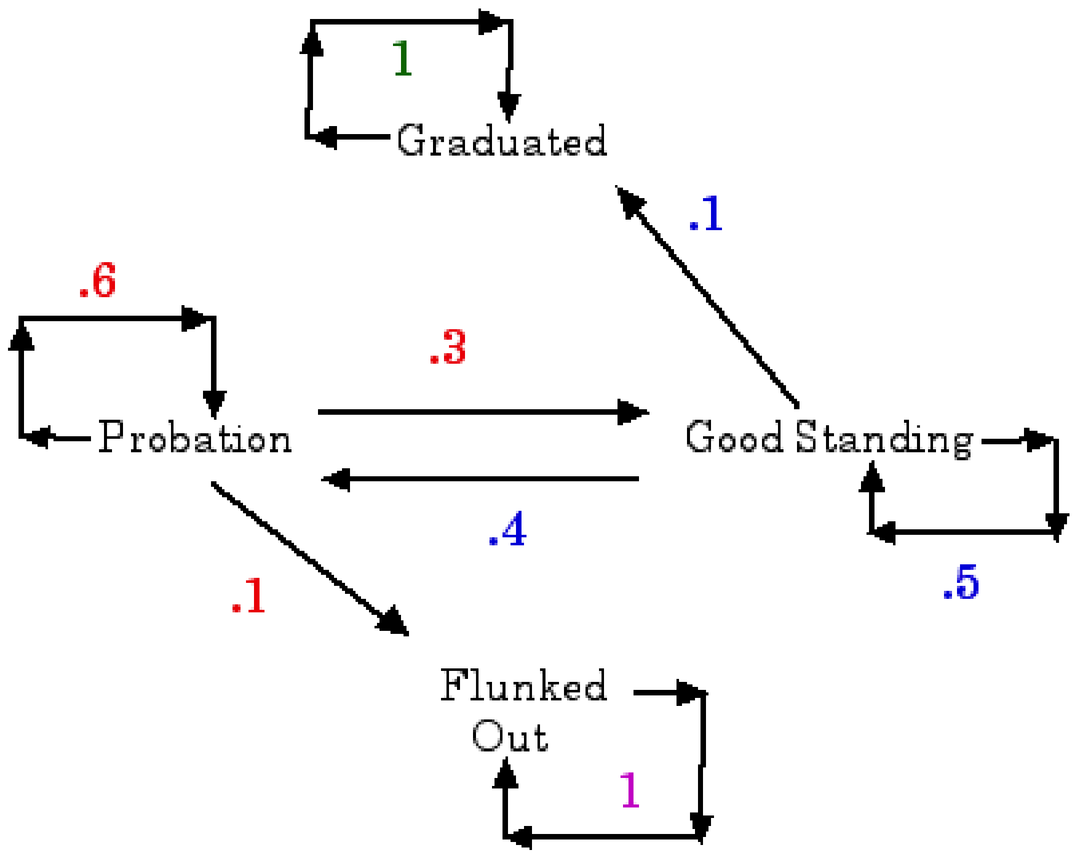
$= p\dfrac{1}{1 - q} = \dfrac{p}{\text{p}} = 1.$

Example 2. Academic Progress Model



Academic Progress Model





The Transition Matrix is

$$\begin{array}{c} \\ F \\ P \\ S \\ G \end{array}\begin{array}{c} \begin{array}{cccc} F & P & S & G \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ .1 & .6 & .3 & 0 \\ 0 & .4 & .5 & .1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Rewrite as

$$\begin{array}{c|cccc|} & F & P & S & G \\ F & 1 & 0 & 0 & 0 \\ P & .1 & .6 & .3 & 0 \\ S & 0 & .4 & .5 & .1 \\ G & 0 & 0 & 0 & 1 \end{array}$$

$$P = \begin{array}{c} \\ F \\ G \\ P \\ S \end{array} \begin{array}{c} \begin{array}{cccc} F & G & P & S \end{array} \\ \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline .1 & 0 & .6 & .3 \\ 0 & .1 & .4 & .5 \end{array}\right) \end{array}$$

which has the structure

$$P = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{R} & \mathbf{Q} \end{pmatrix}$$

Then

$$P^2 = \left(\begin{array}{c|c} I & 0 \\ \hline R & Q \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline R & Q \end{array}\right) = \left(\begin{array}{c|c} I & 0 \\ \hline R + QR & Q^2 \end{array}\right)$$

and

$$P^3 = \left(\begin{array}{c|c} I & 0 \\ \hline R & Q \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline R + QR & Q^2 \end{array}\right)$$

$$= \left(\begin{array}{c|c} I & 0 \\ \hline R + QR + Q^2R & Q^3 \end{array}\right)$$

$$= \left(\begin{array}{c|c} I & 0 \\ \hline (I + Q + Q^2)R & Q^3 \end{array}\right)$$

In general,

$$P^n = \left(\begin{array}{c|c} I & 0 \\ \hline (I + Q + Q^2 + \ldots + Q^{n-1})R & Q^n \end{array}\right)$$

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Let $S = I + Q + Q^2 + Q^3 + \ldots + Q^{n-1}$

So $QS = \quad Q + Q^2 + Q^3 + \ldots + Q^{n-1} + Q^n$

Thus $S - QS = I - Q^n$ *or* $(I - Q)\,S = I - Q^n$

As $n \to \infty$: $(I - Q)\,S \to I$

so $S \to (I - Q)^{-1}$

Hence $P^n \to \left(\begin{array}{c|c} I & 0 \\ \hline (I - Q)^{-1} R & 0 \end{array}\right)$

Absorbing Markov Processes In General

An **Absorbing Markov Chain** has

(a) At least one absorbing state S_i such that $p_{ii} = 1$;

(b) For each nonabsorbing (**transient**) state, it is possible to reach at least one absorbing state in a finite number of steps with positive probability.

Two Basic Questions We Can Answer:

(1) What are the probabilities of winding up in each of the absorbing states?

(2) What is the expected number of steps before absorption?

To Answer: Rewrite the Transition Matrix in Standard Form:

$$\begin{array}{r|cc} & Absorbing: k & Transient: r-k \\ Absorbing: k & I_{k\times k} & 0_{k\times(r-k)} \\ Nonabsorbing: r-k & R_{(r-k)\times k} & Q_{(r-k)\times(r-k)} \end{array}$$

The **Fundamental Matrix** of the Chain is:

$$N = (I - Q)^{-1}$$

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Then

(1) N_{ij} = Average Number of Steps the process is in state j if it begins in state i

(2) t_i = sum of entries in row i of N = expected number of steps until absorbtion if we start in state i

(3) b_{ik} = ikth entry of $B = NR$ = probability of being absorbed in state k if we start in state i.

Our Example:

$$P = \begin{array}{c} \\ F \\ G \\ P \\ S \end{array}\!\!\begin{array}{c} \begin{array}{cccc} F & G & P & S \end{array} \\ \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline .1 & 0 & .6 & .3 \\ 0 & .1 & .4 & .5 \end{array}\right) \end{array}$$

$$\mathrm{N} = (\mathrm{I} - \mathrm{Q})^{-1} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} .6 & .3 \\ .4 & .5 \end{pmatrix}\right]^{-1}$$

$$= \left[\begin{pmatrix} .4 & -.3 \\ -.4 & .5 \end{pmatrix}\right]^{-1} = \begin{pmatrix} \frac{25}{4} & \frac{15}{4} \\ 5 & 5 \end{pmatrix}$$

$$\text{and } \mathbf{B = NR} = \begin{pmatrix} \frac{25}{4} & \frac{15}{4} \\ 5 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{10} \end{pmatrix}$$

$$\mathbf{B} = \begin{array}{c} \\ P \\ S \end{array}\!\!\begin{array}{c} \begin{array}{cc} F & G \end{array} \\ \begin{pmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

$$\mathbf{B} = \begin{array}{c} \\ P \\ S \end{array}\begin{array}{c} \begin{array}{cc} F & G \end{array} \\ \begin{pmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

Conclusions:

Pr(**Eventually Flunking Out** | **On Probation Now**) = 5/8

Pr(**Eventually Flunking Out** | **In Good Standing Now**) = 1/2

Example 3: **Collecting Star Wars Dolls**

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$$\begin{array}{c} \\ None \\ \{Luke\} \\ \{Darth\} \\ Both \end{array} \begin{array}{c} \begin{array}{cccc} None & \{Luke\} & \{Darth\} & Both \end{array} \\ \begin{pmatrix} 0 & p & q & 0 \\ 0 & p & 0 & q \\ 0 & 0 & q & p \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

(1) Rewrite in Standard Form

$$\begin{array}{c} \\ Both \\ None \\ \{Luke\} \\ \{Darth\} \end{array} \begin{array}{c} \begin{array}{cccc} Both & None & \{Luke\} & \{Darth\} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & p & q \\ q & 0 & p & 0 \\ p & 0 & 0 & q \end{pmatrix} \end{array}$$

(2) $Q = \begin{pmatrix} 0 & p & q \\ 0 & p & 0 \\ 0 & 0 & q \end{pmatrix}$ so $I - Q = \begin{pmatrix} 1 & -p & -q \\ 0 & 1-p & 0 \\ 0 & 0 & 1-q \end{pmatrix}$

Finding the inverse of $I - Q = \begin{pmatrix} 1 & -p & -q \\ 0 & 1-p & 0 \\ 0 & 0 & 1-q \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & -p & -q & 1 & 0 & 0 \\ 0 & q & 0 & 0 & 1 & 0 \\ 0 & 0 & p & 0 & 0 & 1 \end{array}\right)$$

$$Row2 = (1/q)Row2$$
$$Row3 = (1/p)Row3$$

$$\left(\begin{array}{ccc|ccc} 1 & -p & -q & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/q & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/p \end{array}\right)$$

$$Row1 = Row1 + pRow2$$
$$Row1 = Row1 + qRow3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & p/q & q/p \\ 0 & 1 & 0 & 0 & 1/q & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/p \end{array}\right)$$

$$N$$

$$\text{Expected Number of Boxes} = \mathbf{1} + \frac{p}{q} + \frac{q}{p}$$