Stochastic Processes I

Probability Models

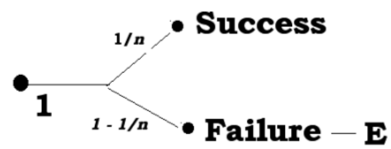
EXPECTED VALUE II

If we have a sequence of independent trials and the probability of a Success on a given trial is *p*, then the Expected Number of trials until we have a first Success is
1/*p*

Example
Trial: Roll a Die
Success: 5
Probability of Success: 1/6
Expected Number of Trials: 6

Method 3

Let E = Expected Number of Trials



$$E = 1 + \left(1 - \frac{1}{n}\right) E$$

$$\Rightarrow E - \left(1 - \frac{1}{n}\right) E = 1$$

$$\Rightarrow E\left(1 - 1 + \frac{1}{n}\right) = 1$$

$$\Rightarrow \left(\frac{1}{n}\right) E = 1$$

$$\Rightarrow E = n$$

Collecting Superheroes

# Collected	*Probability of New Hero in Next Box*	*Waiting Time*
0	***1***	***1***
1	***5/6***	***6/5***
2	***4/6***	***6/4***
3	***3/6***	***6/3***
4	***2/6***	***6/2***
5	***1/6***	***6/1***

Expected Number of Boxes

1 + 6/5 + 6/4 + 6/3 +6/2 + 6/1

$= 6(1/6 + 1/5 + 1/4 + 1/3 + 1/2 + 1)$

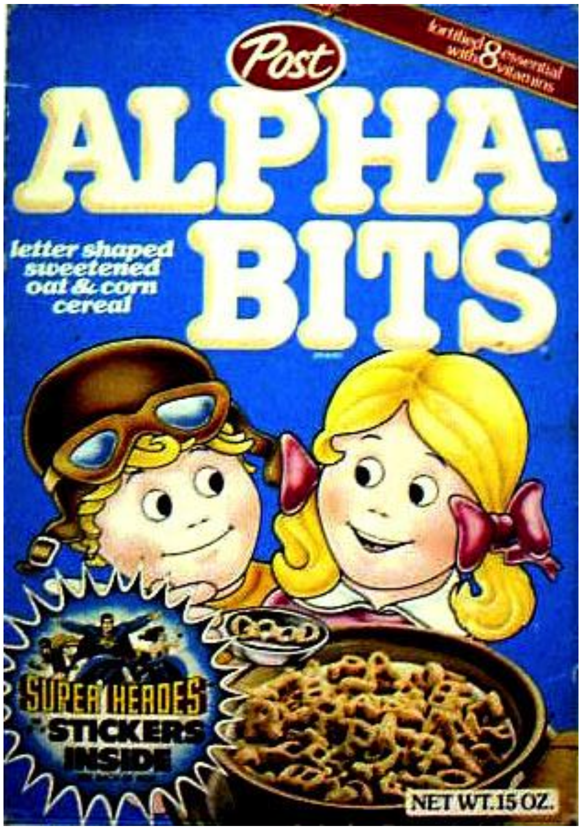
Coupon Collector Problem

n superheroes

Expected Waiting Time:

$n \; (1 + 1/2 + 1/3 + 1/4 + \ldots + 1/n)$

$\sim n \ln n$


Post
fortified with 8 essential vitamins
ALPHA-BITS
letter shaped sweetened oat & corn cereal
SUPER HEROES
STICKERS INSIDE
NET WT. 15 OZ.


ONE SET OF 4
SUPER HEROES
STICKERS INSIDE
BAT MAN
POW
BREAKFAST WITH THE
SUPER HEROES
CONTEST
SUPER HEROES
Viewer
FREE

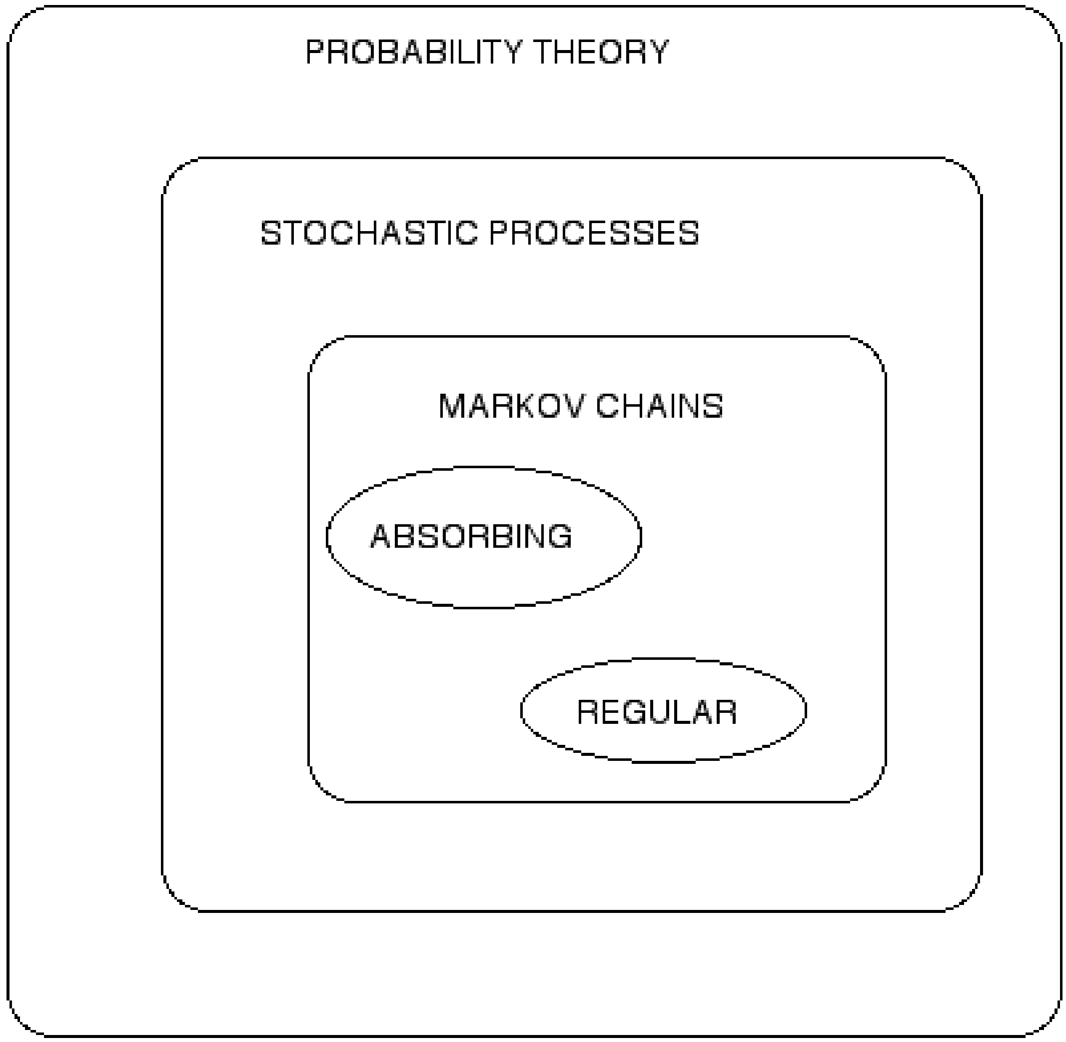
Stochastic Processes

And

Markov Chains

А. А. Марков (1886).

Андрей Андреевич Марков (1856-1922)


PROBABILITY THEORY
STOCHASTIC PROCESSES
MARKOV CHAINS
ABSORBING
REGULAR

Stochastic Process

A **Stochastic Process** is a process that moves in a sequence of ***steps*** through a set of ***states*** such that at each step, there are probabilities of being in each of the possible states.

<u>Examples</u>

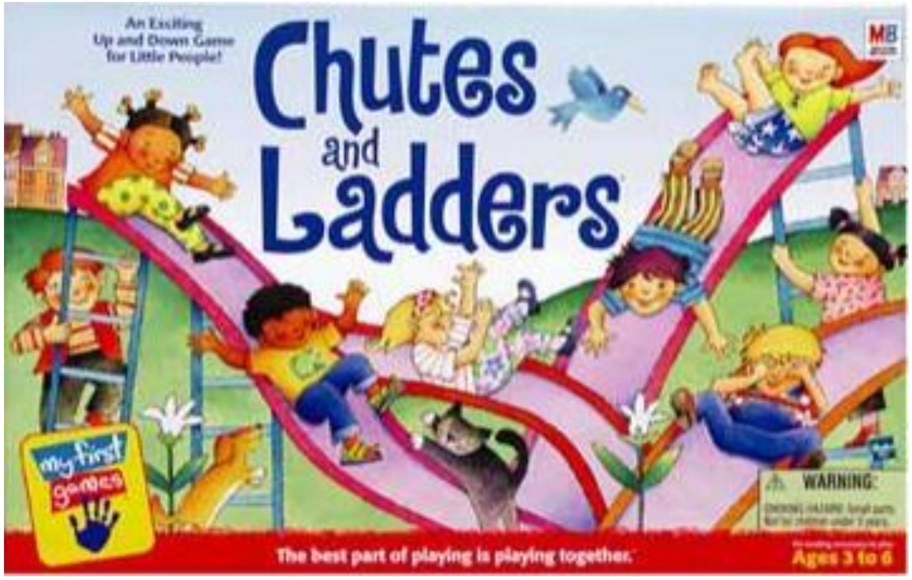
1. The location of the President at noon on successive days.
2. Your GPA at the end of each semester.
2. Number of students arriving on time at each successive meeting of our class.
3. Closing value of the Dow Jones index at the end of each stock exchange session.
4. Democratic membership of the US Senate after each election.
5. Number of victories by Boston Celtics after each game.
6. Random Walk
7. Random Walk with Barriers
8. Outcomes of flip of a coin in a sequence of trials

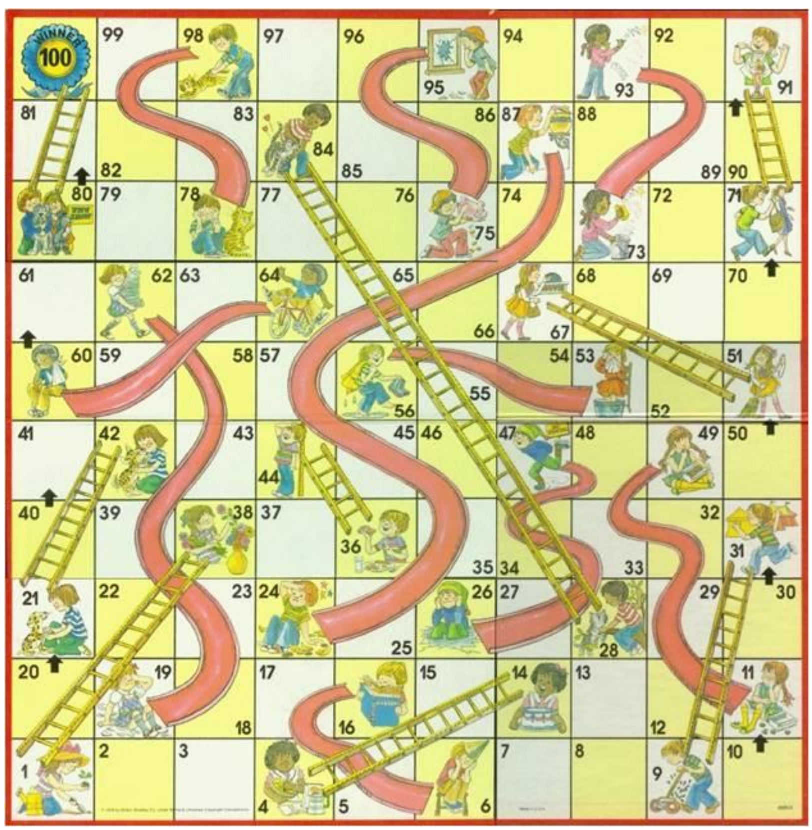
   2 states: Heads, Tails

   Probability of each state is 1/2, and is independent of outcomes on all previous trials.
9. Education level of successive generations of a family.
10. Patient's condition on successive days.
11. Roll a pair of dice and observe the sum.

State	2 or 12	3 or 11	4 or 10	5 or 9	6 or 8	7
Probability	1/36	2/36	3/36	4/36	5/36	6/36

12. Chutes and Ladders


An Exciting
Up and Down Game
for Little People!
Chutes and Ladders
MB
my first games
WARNING:
The best part of playing is playing together.
Ages 3 to 6


WINNER
100

More Examples

Flipping Coins I

Steps: Successive Flips
States: Outcomes of single flip = { *Head, Tail* }

Probability (State is *Heads* at Step 15) = 1/2

Flipping Coins II

Steps: Successive Flips
States: Number of Heads we have flipped = {0,1,2,...}

$$\text{Probability(6 Heads at Step 15)} = \binom{15}{6}\left(\frac{1}{2}\right)^{6}\left(\frac{1}{2}\right)^{9}$$

Steps: Successive Games

States:

(a) Outcome of Game: Win, Lose
(b) Total victories so far
(c) Number of players injured

MARKOV PROCESS

Suppose we have a stochastic process with

- A finite number of states
- The probability of being in a particular state at the *nth* step **depends only on the state occupied at the preceding step**

Example: Social Mobility

Steps: generations

States: Professional, Skilled, Unskilled

		Children	*(generation*	*n+1)*
		P	***S***	***U***
Parents	***P***	.8	.1	.1
generation	***S***	.2	.6	.2
n	***U***	.25	.25	.5

TRANSITION MATRIX

Markov Process

(a) States $S_1, S_2, \ldots, S_r$ and Steps

(b) Transition Probabilities

p_{ij} = **probability of moving from State i to State j in a single step**

$p_{ij} = \Pr[\, S_j \text{ at step}(n+1) \mid S_i \text{ at step } n\,]$

Note: Each $p_{ij} \geq 0$

$$\sum_{j=1}^{r} p_{ij} = 1 \text{ for all } i$$

p_{ij}**'s are independent of** n

(c) Initial Distribution

Two Special Types of Markov Processes

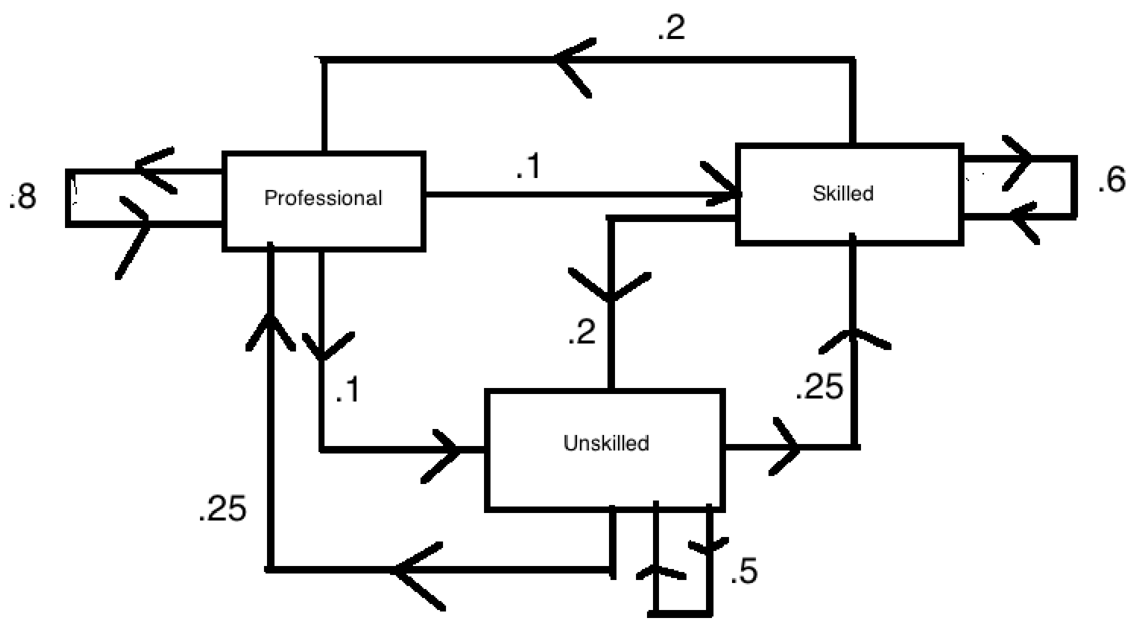
Regular
Absorbing

Social Mobility Example

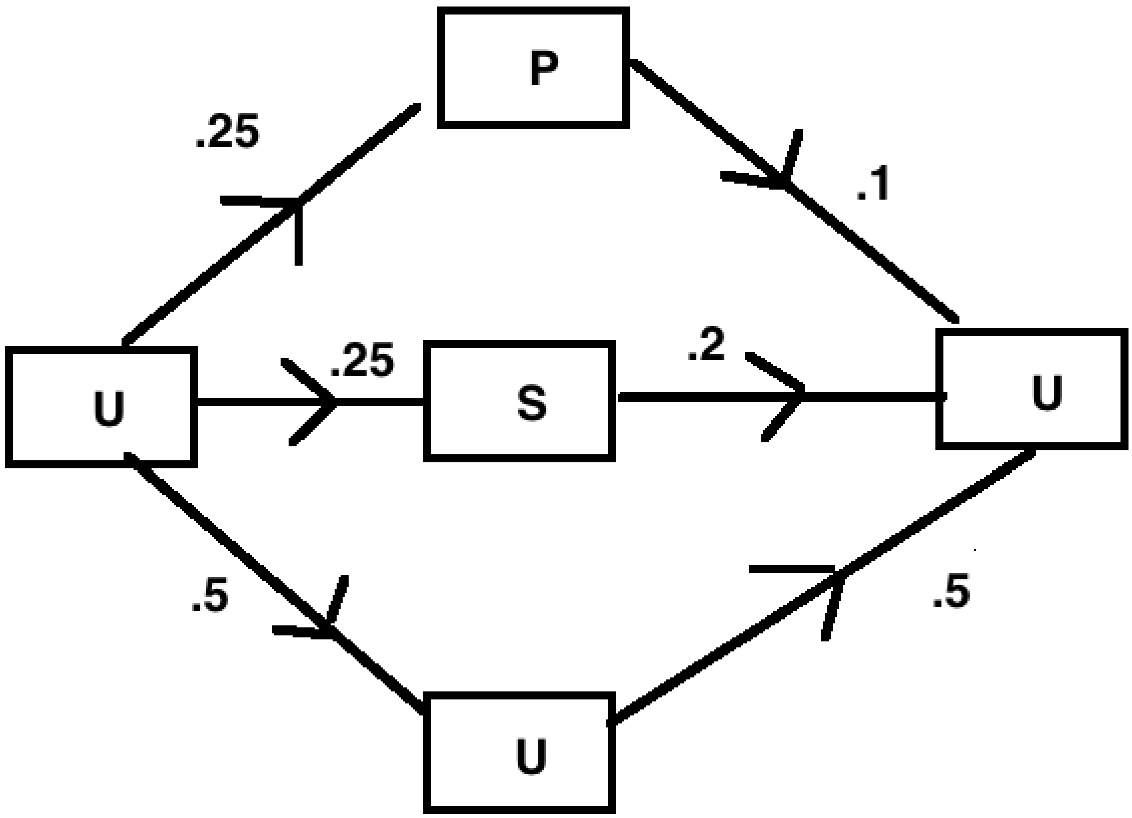
A representation of transition matrix ***P***

		generation (n+1) [children]		
		P	S	U
generation n	P	***.8***	***.1***	***.1***
[parents]	S	***.2***	***.6***	***.2***
	U	***.25***	***.25***	***.5***

State Diagram

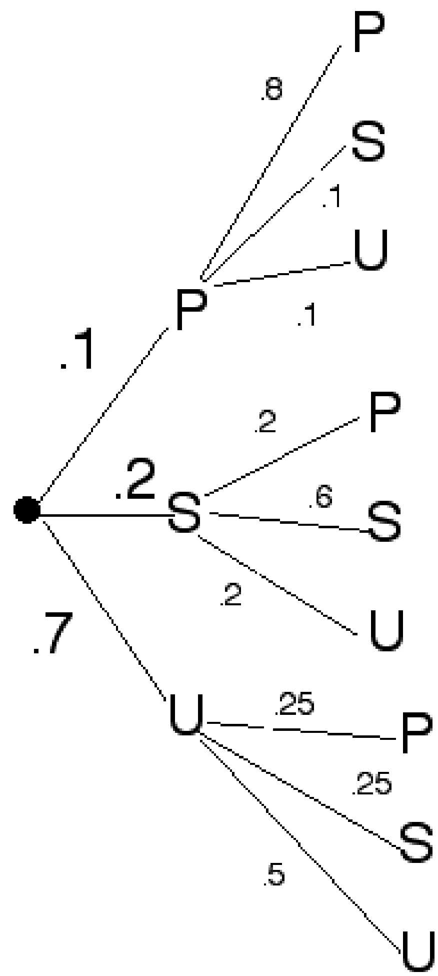


Tree Diagram: Probability that grandchild of unskilled will be unskilled.



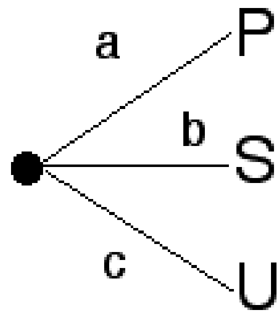
$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ .25 & .25 & .5 \end{bmatrix} \begin{bmatrix} \cdot & \cdot & .1 \\ \cdot & \cdot & .2 \\ \cdot & \cdot & .5 \end{bmatrix}$$

Suppose $\vec{p}^{(0)} = (.1,.2,.7)$. What are $\vec{p}^{(1)}$ and $\vec{p}^{(2)}$, the distributions after 1 and 2 generations?



P	S	U
(.1)(.8)=.08	(.1)(.1) = ..01	(.1)(.1)) = .01
(.2)(.2)=-.04	(.2)(.6) = .12	(.2)(.2) = .04
(.7)(.25)=.175	(.7)(.25)=.175	(.7)(.5)=.35
.295	**.305**	**.40**

Arbitrary initial vector $\vec{p}^{(0)} = (a, b, c)$



$$P = a(.8) + b(.2) + c(.25)$$
$$S = a(.1) + b(.6) + c(.2)$$
$$U = a(.1) + b(.2) + c(.5)$$

so $\vec{p}^{(1)} = \vec{p}^{(0)} P$

2nd generation:

$$\vec{p}^{(2)} = \vec{p}^{(1)} P = (\vec{p}^{(0)} P) P = \vec{p}^{(0)} P^2$$

For a Markov Process with transition matrix $\boldsymbol{P}$, let $\vec{p}^{(k)}$ be the probability distribution after k steps; that is, the ith component of $\vec{p}^{(k)}$ is the probability of being in state i after k steps.

Lemma:

$$\boldsymbol{p}^{(n+1)} = \boldsymbol{p}^{(n)}\boldsymbol{P}$$

Theorem:

$$\boldsymbol{p}^{(n)} = \boldsymbol{p}^{(0)}\boldsymbol{P}^{n}$$

$$p_i^{(n+1)} = \sum_{k=1}^{n} p_k^{(n)} p_{ki} = [\,\overrightarrow{\mathbf{p}^{(\mathbf{n})}}\boldsymbol{P}\,]_i$$

What is the Long Term Behavior?

$$\lim_{n\to\infty} \vec{p}^{(n)} = \vec{p}^{(0)} \lim_{n\to\infty} P^n$$

$$P = \begin{pmatrix} .8 & .1 & .1 \\ .2 & .6 & .2 \\ .25 & .25 & .5 \end{pmatrix}$$

Run the program MARKOV to test: