Stochastic Processes
and
Markov Chains II

(1856-1922)

Markov Process

(a) ***States*** **$S_1, S_2, \ldots, S_r$ and Steps**

(b) ***Transition Probabilities***

p_{ij} = probability of moving from State i to State j in a single step

p_{ij} = Pr[S_j at step(n+1) | S_i at step n]

Note: Each $p_{ij} \geq 0$

$$\sum_{j=1}^{r} p_{ij} = 1 \text{ for all } i$$

p_{ij}'s are independent of n

(c) ***Initial Distribution*** $\overrightarrow{p^{(o)}}$

Two Special Types of Markov Process

REGULAR
Some Power of the Transition Matrix Has All Positive Entries

ABSORBING

Social Mobility Example

A representation of transition matrix ***P***

generation (n+1) [children]

		P	S	U
generation n	P	***.8***	***.1***	***.1***
[parents]	S	***.2***	***.6***	***.2***
	U	***.25***	***.25***	***.5***

An Absorbing Matrix

step (n+1)

step n		**A**	*B*	*C*
	A	***1***	***0***	***0***
	B	***.2***	***.6***	***.2***
	C	***0***	***. 5***	***.5***

A is an ***Absorbing*** State

B, C are Transient States

What is the Long Term Behavior?

$$\lim_{n\to\infty} \vec{p}^{(n)} = \vec{p}^{(0)} \lim_{n\to\infty} P^n$$

$$P = \begin{pmatrix} .8 & .1 & .1 \\ .2 & .6 & .2 \\ .25 & .25 & .5 \end{pmatrix}$$

Run the program MARKOV to test:

$\vec{p}^{(n)}$ seems to approach (.53, .26, .21)

Can we find $\vec{w} = \lim_{n\to\infty} \vec{p}^{(n)}$ analytically?

$\vec{w} = \vec{w}P$ where $\vec{w} = (x, y, z)$ and $x + y + z = 1$

$$(x, y, z) = (x, y, z)\begin{pmatrix} .8 & .1 & .1 \\ .2 & .6 & .2 \\ .25 & .25 & .5 \end{pmatrix}$$

$$\begin{cases} x = .8x + .2y + .25z \\ y = .1x + .6y + .25z \\ z = .1x + .2y + .5z \end{cases}$$

$$(1) \qquad -.2x + .2y + .25z = 0$$

$$(2) \qquad 1x - .4y + .25z = 0$$

$$(3) \qquad .1x + .2y - .5z = 0$$

Rewriting (1) and (2):

$$\begin{matrix} .25z = .2x - .2y \\ .25z = -.1x + .4y \end{matrix} \Rightarrow .2x - .2y = -.1x + .4y$$

$$\Rightarrow .3x = .6y \Rightarrow y = \frac{1}{2}x$$

Substitute in (3):

$$.1x + .2(.5x) - .5z = 0 \Rightarrow .2x = .5z \Rightarrow z = \frac{2}{5}x$$

Then $x + y + z = 1$

becomes $x + \frac{1}{2}x + \frac{2}{5}x = 1$

$$10x + 5x + 4x = 10$$

$$19x = 10$$

$$x = 10/19$$

$$y = 5/19$$

$$z = (2/5)(10/19) = 4/19$$

long term: $\left(\frac{10}{19}, \frac{5}{19}, \frac{4}{19}\right) \approx (.526,\ .263,\ .211)$

Main Result on Regular Markov Chains and Examples

Definition. A Markov Process is **regular** if some positive power of the transition matrix has all positive entries.

Example.

$$P = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \Rightarrow P^2 = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

Theorem: If P is the transition matrix of a regular Markov Process, then there is a vector $\mathbf{w}$ all of whose components are strictly positive such that

(1) $\mathbf{w} = \lim_{n\to\infty} \mathbf{p}^{(\mathbf{n})}$

(2) $\mathbf{w}$ is independent of $\mathbf{p}^{(\mathbf{0})}$

(3) $\mathbf{w}$ is the unique probability vector satisfying $\mathbf{w} = \mathbf{w}\,P$

(4) $$\lim_{n\to\infty} P^n = \begin{pmatrix} \mathbf{w} \\ \mathbf{w} \\ \cdots \\ \cdots \\ \mathbf{w} \end{pmatrix}$$

(5) $\mathbf{w}$ is a left eigenvector of P associated with eigenvalue 1

Example : If **w** = (*x,y),* then

$$(x,y)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (x,y) \Rightarrow \begin{matrix} x = 0x + \frac{1}{2}y \\ y = 1x + \frac{1}{2}y \end{matrix} \Rightarrow \begin{matrix} y = 2x \\ y = 2x \end{matrix}$$

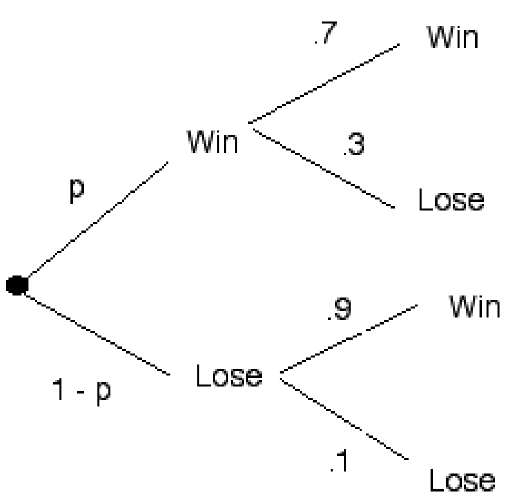
but $x + y = 1$ so $1 - x = 2x$ so $x = 1/3$

and $\mathbf{w} = \left(\frac{1}{3}, \frac{2}{3}\right)$

Check: $\left(\frac{1}{3}, \frac{2}{3}\right)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{1}{3}, \frac{1}{3} + \frac{1}{3}\right)$

A Baseball Example

		game (n+1)	
		Win	*Lose*
game n	**Win**	*.7*	*.3*
	Lose	*.9*	*.1*



$$= \left(.7p + .9(1-p), .3p + .1(1-p)\right)$$

$$\mathbf{p}^{(2)} = (.72 + .04p, .28 - .04p)$$

$$\mathbf{p}^{(1)} = \left(.7p + .9(1-p), .3p + .1(1-p)\right)$$

$$= (.9 - .2p, .1 + .2p)$$

$$\mathbf{p}^{(2)} = (.72 + .04p, .28 - .04p)$$

Can check by Induction that $\mathbf{p}^{(n)} = \left(\frac{3}{4} + \left(\frac{-1}{5}\right)^n (p - \frac{3}{4}), \frac{1}{4} - \left(\frac{-1}{5}\right)^n (p - \frac{3}{4})\right)$ so

$$\lim_{n\to\infty} \mathbf{p}^{(n)} = \left(\frac{3}{4}, \frac{1}{4}\right)$$

Solving by $\mathbf{w} = \mathbf{w}P$ gives $(x, y)\begin{pmatrix} .7 & .3 \\ .9 & .1 \end{pmatrix} = (x, y)$ or $(x, y) = (.7x + .9y, .3x + .1y)$

$$\begin{matrix} x = .7x + .9y \\ y = .3x + .1y \end{matrix} \Rightarrow \begin{matrix} -.3x = .9y \\ .9y = .3x \end{matrix} \Rightarrow y = \frac{x}{3}$$ but $x + y = 1$ so $x + \frac{x}{3} = 1$ or $x = \frac{3}{4}$

Eigenvalue/Eigenvector Analysis

$$P = \begin{bmatrix} \frac{7}{10} & \frac{3}{10} \\ \frac{9}{10} & \frac{1}{10} \end{bmatrix} \Rightarrow P - \lambda I = \begin{bmatrix} \frac{7}{10} - \lambda & \frac{3}{10} \\ \frac{9}{10} & \frac{1}{10} - \lambda \end{bmatrix}$$

$$\text{so} \quad \det(P - \lambda I) = \left(\tfrac{7}{10} - \lambda\right)\left(\tfrac{1}{10} - \lambda\right) - \left(\tfrac{9}{10}\right)\left(\tfrac{3}{10}\right)$$

and $\det(P - \lambda I) = 0$ becomes

$$\lambda^2 - \frac{8}{10}\lambda - \frac{20}{100} = 0$$

$$\Rightarrow 5\lambda^2 - 4\lambda - 1 = 0 \Rightarrow (5\lambda + 1)(\lambda - 1) = 0 \text{ so } \lambda = -\frac{1}{5} \text{ or } \lambda = 1$$

Since there are 2 distinct eigenvalues, there is a 2 x 2 invertible matrix S such that

$$S^{-1}PS = \begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{5} \end{bmatrix} \text{ so that } P = S\begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{5} \end{bmatrix}S^{-1} \text{ and hence}$$

$$P^n = S\begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{5} \end{bmatrix}^n S^{-1} = S\begin{bmatrix} 1^n & 0 \\ 0 & (\frac{-1}{5})^n \end{bmatrix}S^{-1} = S\begin{bmatrix} 1 & 0 \\ 0 & (\frac{-1}{5})^n \end{bmatrix}S^{-1}$$

The matrix S can be formed by taking right eigenvectors associated with the eigenvalues as its columns. An eigenvector associated with $\lambda = 1$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and an eigenvector associated with $\lambda = -\frac{1}{5}$ is $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$. Thus we can let $S = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$. T hen $S^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{-1}{4} \end{bmatrix}$.

And det(P - λI) = 0 becomes

$$\lambda^2 - \frac{8}{10}\lambda - \frac{20}{100} = 0$$

$$\Rightarrow 5\lambda^2 - 4\lambda - 1 = 0 \Rightarrow (5\lambda + 1)(\lambda - 1) = 0$$

so $\lambda = -\frac{1}{5}$ or $\lambda = 1$

Since there are 2 distinct eigenvalues, there is a 2 x 2 invertible matrix S such that

$S^{-1}PS = \begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{5} \end{bmatrix}$ so that $P = S\begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{5} \end{bmatrix}S^{-1}$ and hence

$$P^n = S\begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{5} \end{bmatrix}^n S^{-1} = S\begin{bmatrix} 1^n & 0 \\ 0 & (\frac{-1}{5})^n \end{bmatrix} S^{-1} = S\begin{bmatrix} 1 & 0 \\ 0 & (\frac{-1}{5})^n \end{bmatrix} S^{-1}$$

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$$S^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

Thus $P^n = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (\frac{-1}{5})^n \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{-1}{4} \end{bmatrix} =$

$$\begin{bmatrix} 1 & (\frac{-1}{5})^n \\ 1 & -3(\frac{-1}{5})^n \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{-1}{4} \end{bmatrix}$$ so we have

$$P^n = \begin{bmatrix} \frac{3}{4} + \frac{1}{4}(\frac{-1}{5})^n & \frac{1}{4} - \frac{1}{4}(\frac{-1}{5})^n \\ \frac{3}{4} - \frac{3}{4}(\frac{-1}{5})^n & \frac{1}{4} + \frac{3}{4}(\frac{-1}{5})^n \end{bmatrix}$$