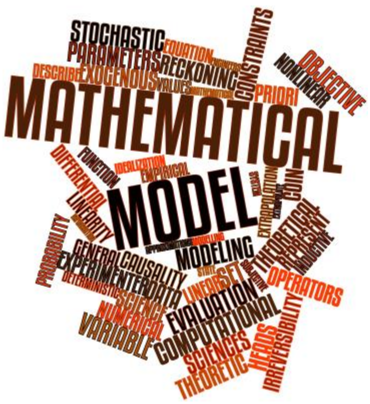
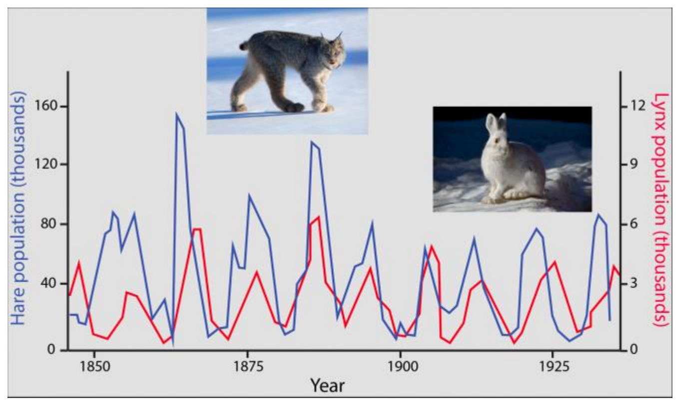
MATHEMATICAL MODELING
MATH 315





Fall 2024

Professor Michael Olinick

MATHEMATICAL MODELS

FALL 2024

HANDOUTS:

(1) Course Description

(2) Assignments 0 and 1

(3) “A Biologist’s Mathematics” by David Causton

(4) Exploratory 1: Computer Modeling of Plant Growth

(5) Questionnaire/Course Schedule

MATH 315 Mathematical Modeling Fall. 2024

Student Information Questionnaire

Name:______________________________ Year of Graduation: ________

Home City & State or Country:

____

Telephone: _____________________________Major or Probable Major: ___
____

Are you interested in completing the Applied Math Track within the Mathematics Major? ____________

Mathematics Background

Previous Mathematics Courses:

Other related courses (Physics, Chemistry, Economics):

How much experience have you had with *Maple, Mathematica* or *MATLAB?*

What are your reasons for studying Calculus?

Please fill in your class schedule on the reverse side.

	Monday	Tuesday	Wednesday	Thursday	Friday

50 Minute Classes					
8:40 – 9:30					
9:45 – 10:35					
11:15 – 12:05	**MATH 315**		**MATH 315**		**MATH 315**
1:10 – 2:00					
2:15 – 3:05					
3:20 – 4:10					
75 Minute Classes					
8:15 – 9:30					
9:45 – 11:00					
11:15 - 12:30					
12:45 – 2:00					
2:15 – 3:30					
7:30– 8:45PM					
Other					
1:30 – 4:15					
7:30 – 10:30					

MATH 315: *Mathematical Modeling*
Course Description
Fall Term 2024

Course Title: Mathematical Modeling

Course Description: An introduction into the process of developing and interpreting mathematical models within the framework of numerous applications. We will utilize discrete, continuous, and probabilistic approaches to explore applications in a wide range of fields with an emphasis on mathematical models of proven usefulness in problems arising in the life and social sciences. We will explore specific models in population dynamics, epidemiology, ecology, political science, ecology, sociology, anthropology, psychology, and economics. We will use MATLAB to implement and analyze several of these models. If time permits, we will examine simulation models using NETLOGO. *Mathematical Modeling* is a core course in the Applied Mathematics track of the Mathematics major; it is a prerequisite for the Senior Seminar MATH 715 to be offered in Spring 2026.

Prerequisites: MATH 200 (*Linear Algebra*) and MATH 226 (*Differential Equations*) or by instructor's approval.

CW Option: Students interested in using the course to satisfy the college's writing requirement should sign up for the CW section (MATH 315B).

Instructor: Michael Olinick, 202 Warner, Phone: 443-5559. Home telephone: 388-4290; email: *molinick@middlebury.edu*. Usual Office Hours: Monday and Wednesday: 9:30 –11 AM and 12:10 - 1 PM; Thursday: 10 AM – Noon, and Friday: 9:30 – 11 AM. **I am happy to make an appointment to see you at other mutually convenient times.**.

Meeting Times: MWF 11:15 AM – 12:05 PM, Warner 11.

Textbooks: The basic text will be my book **Mathematical Modeling in the Social and Life Sciences** (Wiley, 2014). There will be a copy of this book, as well as the Brannan and Boyce *Differential Equations* text, on reserve in the Davis Family Library. In addition, there will be many additional notes distributed as well as possibly some readings on reserve in the library.

Additional Course Materials: See the course webpage at http://f24.middlebury.edu/math0315a and/or the course folder on the file server **Classes/Fall2024/math0315a.**

Requirements: There will be one evening mid-term examination and a final examination in addition to required daily homework assignments and an independent project. Ideally, the project would be the creation, analysis and testing of a mathematical model of a real world problem of interest to the student, but it might consist of more extended reading and problem solving or a critical review of some of the literature in mathematical model building.

The various components of the course work and their approximate weight in determining a final grade will be these:

A. Homework Assignments/ Lab Reports [25% of course grade]

B. Mid-term Examination (Monday Evening, October 21) [25%]

C. Final Examination [30%] (**9 AM – Noon, Wednesday, December 11**)

D. Independent Project [20%]. This project is due no later than Friday, December 6; this is a very firm deadline. More information about this project will be distributed next week.

<u>*Comments*</u>: Topics that I would like to cover include arms race models, population dynamics, mathematical ecology, epidemic modeling, cultural stability, criminal justice systems, residential segregation, opera and the Bible. Mathematical techniques that will be introduced will include differential equations, autonomous systems, Markov processes, game theory and computer simulation.

Extensive use will be made of MATLAB. We will also introduce NetLogo, the premiere software tool for agent based modeling.

Accommodations: Students who have Letters of Accommodation in this class are encouraged to contact me as early in the semester as possible to ensure that such accommodations are implemented in a timely fashion. For those without Letters of Accommodation, assistance is available to eligible students through Student Accessibility Services. Please contact Jodi Litchfield or Courtney Cioffredi, the ADA Coordinators, for more information: Courtney Cioffredi can be reached at ccioffredi@middlebury.edu or 802-443-2169 and Jodi Litchfield can be reached at litchfie@middlebury.edu or 802-443-5936. All discussions will remain confidential.

> People who are offended by strong language should skip this note. I am going to use the strongest language I know, apart from poetry, which is mathematics. In fact, its terseness, immense analogical power and frequent difficulty make mathematics the poetry of the sciences. If you haven't read some mathematics or some poetry lately you're not having as much fun in life as you could be.
>
> -Joel Cohen, *How Many People Can The Earth Support*

MATH 315: Fall, 2024

Further Course Information

Tentative Course Outline (Time may not permit covering all topics; some substitutions may occur to reflect student/faculty interests)

I. Introduction: What is a Mathematical Model?
- A. The Analytical Approach
- B. The Simulation Approach

II. Deterministic Models
- A. Richardson's Arms Race Model
- B. Population Dynamics
  - 1. Single Species
  - 2. Interacting Species: Predation and Competition
- C. Deterministic Epidemic Models

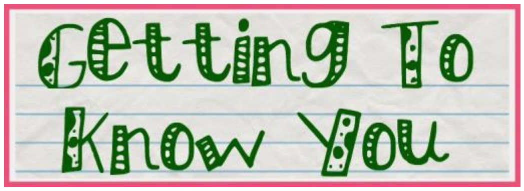
III. Probabilistic Models
- A. Markov Chains
- B. Hoffman's Models of Cultural Stability
- C. Blumstein-Larson Models of Recidivism in Criminal Justice System
- D. Stochastic Epidemic Models

IV. Models of Decision Making Under Uncertainty or Conflict
- A. Models of Economic and Social Justice
- B. Game Theory Models of Old Testament Stories
- C. Prisoner's Dilemma and *Tosca*
- D. Evolutionary Game Theory

VI. Computer Simulation Models
- A. A Hospital Planning Model
- B. Agent-Based Modeling and Residential Segregation

VII. A Deeper Dive into Differential Equations
- A. Existence – Uniqueness Theorems
- B. Poincare – Bendixson Theorem

MATH 315 *Mathematical Modeling* Fall 2024
Assignment 0
(Adapted from *Homework 0* by Alex Lyford)
Due: Wednesday, September 11



Reading

Read carefully through the materials in the First Day Packet.

Writing

You may submit an electronic copy of this assignment to me (molinick@middlebury.edu) with the subject line: **MATH 122 Assignment 0** or print it out and bring it to Wednesday's class. Make sure you include your name at the top of the document.

Your task is to create a document describing yourself, your goals, and what you hope to get out of our Differential Equations class. Please provide your name at the top of the first page along with your major or likely major and your anticipated graduate date.

Start with an autobiographical statement about yourself that will help me to get to know a little about you. Where did you grow up? Why did you come to Middlebury? What are your likes and dislikes? Do you have any hobbies that you do regularly? Do you have a major extracurricular activity such as athletics, theatre or *The Campus?*

After the biographical statement tell me about your mathematical, statistical, and computer programming background. What did you like about previous mathematics, statistics, and/or programming classes? What did you dislike? What aspects did you find easy? What aspects did you find challenging?

The next part should discuss your plans for the remainder of your time in college, and what you hope to do after you graduate. Is more schooling the next step, or do you plan to get a job? It's okay to not have any idea what you want to do after graduation, but list some possibilities so that I can better tailor the materials in class to your potential career opportunities.

Finally tell me about your thoughts and expectations for this class. What are you hoping and/or expecting to learn? What do you think the challenges of this course might be? What, if anything, have you heard about this course from your peers? What expectations do you have of me? Feel free to also discuss anything I've failed to ask here!

MATH 315: Fall 2024

Assignment 1

Due: Friday, September 17

Quote of the Day

The idea that something – anything – could be known with certainty... was delightful, intoxicating, especially when ... it opened up the possibility that other things too might be amenable to strict, mathematical proof. Perhaps even disputes between people might be resolved in this way. 'I hoped,' Russell wrote in *Portraits from Memory*, 'that in time there would be a mathematics of human behavior as precise as the mathematics of machines.'

- Ray Monk, *Bertrand Russell The Spirit of Solitude*

I. **Mathematical Models**.

Read Preface and Chapter 1 of our text by this Wednesday.

Read the excerpts from David R. Causton, *A Biologist's Mathematics,* London: Edward Arnold, 1977.

View also: Harpo Marx - The Story Of Mankind at **http://www.youtube.com/watch?v=H7de1sTeD6w**
Write up solutions for Exercises 10, 12 - 16, 26, and 27 in Chapter 1 and the problem on *Vertical Motion with a Retarding Force* on the next page. Although this problem has many parts, each one only involves material from single variable calculus and some algebraic manipulations. The link Solving y' = ay+ b on our course website may be useful.

For Exercise 10, you may wish to consider some of your favorite similes or metaphors. Here are a couple of famous ones:

"A dream deferred dries up like a raisin in the sun" - Langston Hughes

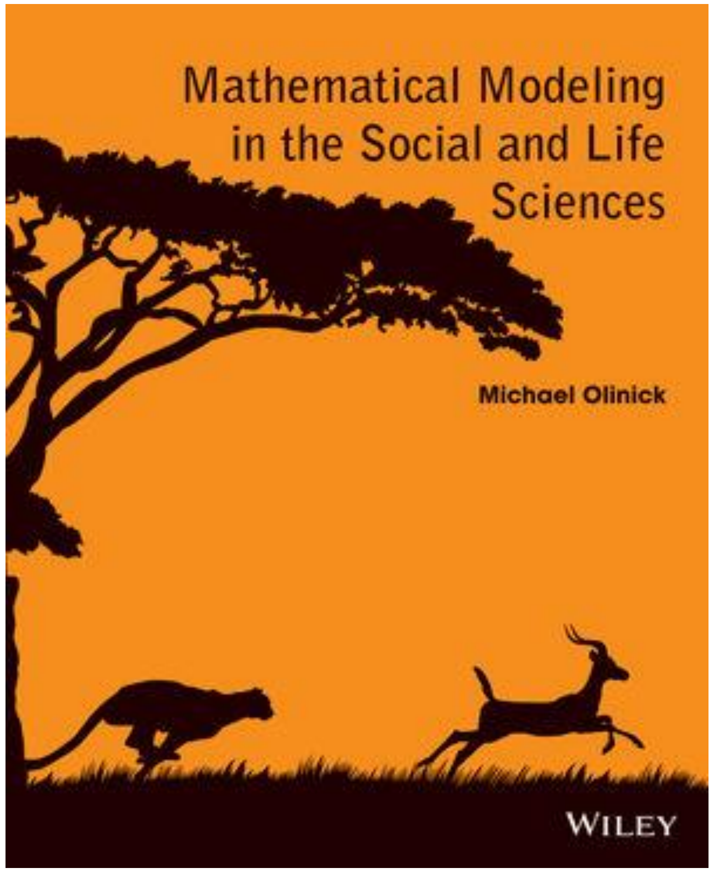
"The righteous shall flourish like the palm tree,
and grow mighty like a cedar in Lebanon.
Planted in the house of the Lord,
They shall flourish in the courts of our God.
Even in old age they shall bring forth fruit,
They shall be full of vigor and strength."
- Psalm 92

You may also wish to comment on the confusion caused by such clumsy "mixed metaphors" as *Milking the migrant workers for all they were worth, the supervisors barked orders at them.*

Vertical Motion With a Retarding Force

Suppose we project a ball of mass *m* vertically upward from the Earth's surface with a positive initial velocity v_0. Let *y(t)* represent the height of the ball above the surface at time *t* while *v(t)* represents its velocity.
Assume that the gravitational force *g* is constant and that there is a retarding force (e.g., friction, air resistance) opposite to the direction of motion and having magnitude proportional to the velocity.

1) Show that in both the ascent and descent of the ball, the total force acting on the ball at time t is
$-mg - p\,v(t)$ where p is a positive constant.
2) Explain why Newton's Second Law of Motion implies
$m\,v'(t) = -mg - p\,v(t)$
3) Solve the differential equation and show
$$v(t) = \left(v_0 + \frac{mg}{p}\right) e^{\left(-\frac{p}{m}t\right)} - \left(\frac{mg}{p}\right)$$
4) Integrate the expression for *v(t)* to find *y(t)* as an explicit function of *t.*
5) Let t* be the time it takes the ball to ascend to its maximum height where its velocity is 0. Show that
$$t^* = \frac{m}{p}\ ln\left(\frac{mg + pv_o}{mg}\right)$$
6) Explain why $y(2t^*) > 0$ implies that the time of descent exceeds the time of ascent; that is, the ball spends more time falling than rising.
7) Find $y(2t^*)$ as an explicit function of *t*.
8) Let $x = e^{\left(\frac{p}{m}t^*\right)}$ and show $x = \frac{mg+pv_0}{mg}$
9) Explain why we know *x* > 1.
10) Using your result from (7), show $y(2t^*) = \frac{m^2 g}{p^2}\left(x - \frac{1}{x} - 2\ln x\right)$
11) Consider the function $F(x) = x - \frac{1}{x} - 2\ln x$ for $x \geq 1$. Sketch a graph of *F* and use its derivative F' to give a convincing argument that *F* is a strictly increasing function.
12) Explain why $y(2t^*)$ is positive and hence the time of ascent is less than the time of descent.
13) (*Optional Extra Credit*): The assumption that the retarding force is proportional to velocity is a reasonable one in some situations, but fails in others. The drag on a skydiver or parachutist is better approximated as proportional to the square of the velocity, for example. The drag on a golf ball (because of spin) is more accurately assumed to be proportional to $v^{1.3}$.
Suppose all we know about the retarding force – *f(v)* is that it is a continuously differentiable function of velocity that satisfies
$v\,f(v) > 0$ for $v \neq 0$;
that is, *f(v)* > 0 if *v* > 0 and *f(v)* < 0 if *v* < 0.
Show that the time of ascent is less than the time of descent.

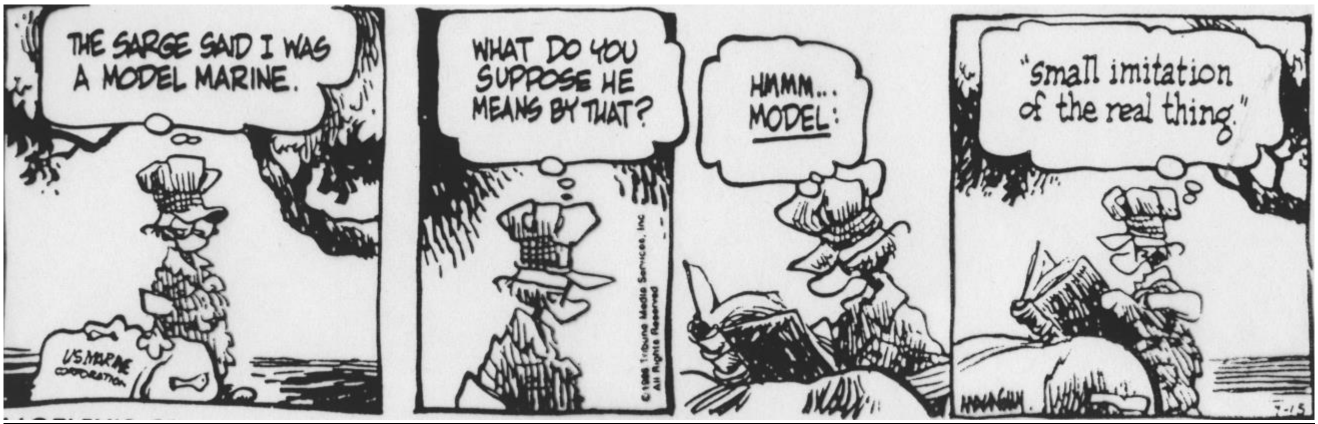

Mathematical Modeling
in the Social and Life
Sciences
Michael Olinick
WILEY

WHAT

IS

A

MODEL?



THE SARGE SAID I WAS A MODEL MARINE.
U.S. MARINE CORPORATION
WHAT DO YOU SUPPOSE HE MEANS BY THAT?
© 1988 Tribune Media Services, Inc
All Rights Reserved
HMMM... MODEL:
"small imitation of the real thing."

WHAT IS A MODEL?

The object of study in all sciences is the Real World

The aims of the study:

(1) **Discover** the laws that govern behavior

(2) **Predict** what future behavior will be

(3) **Control** (influence) future behavior.

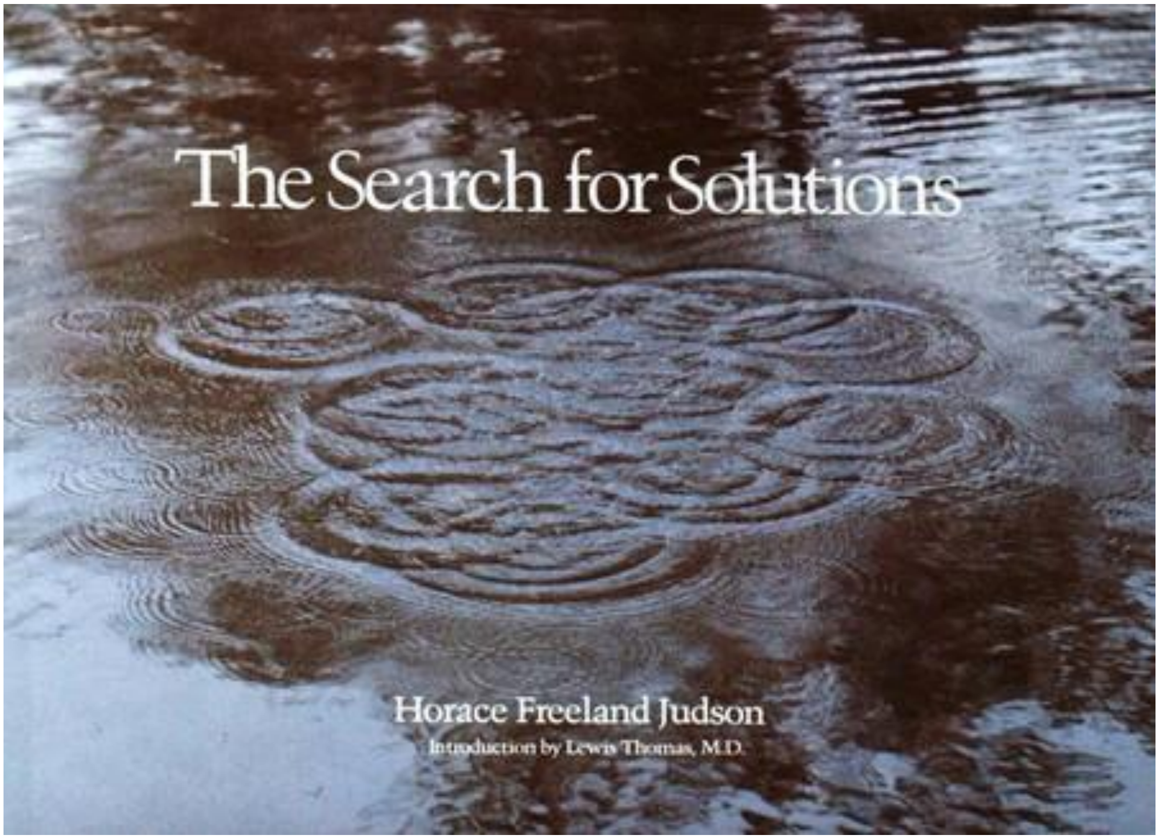
What is a model and how does it fit into this study?

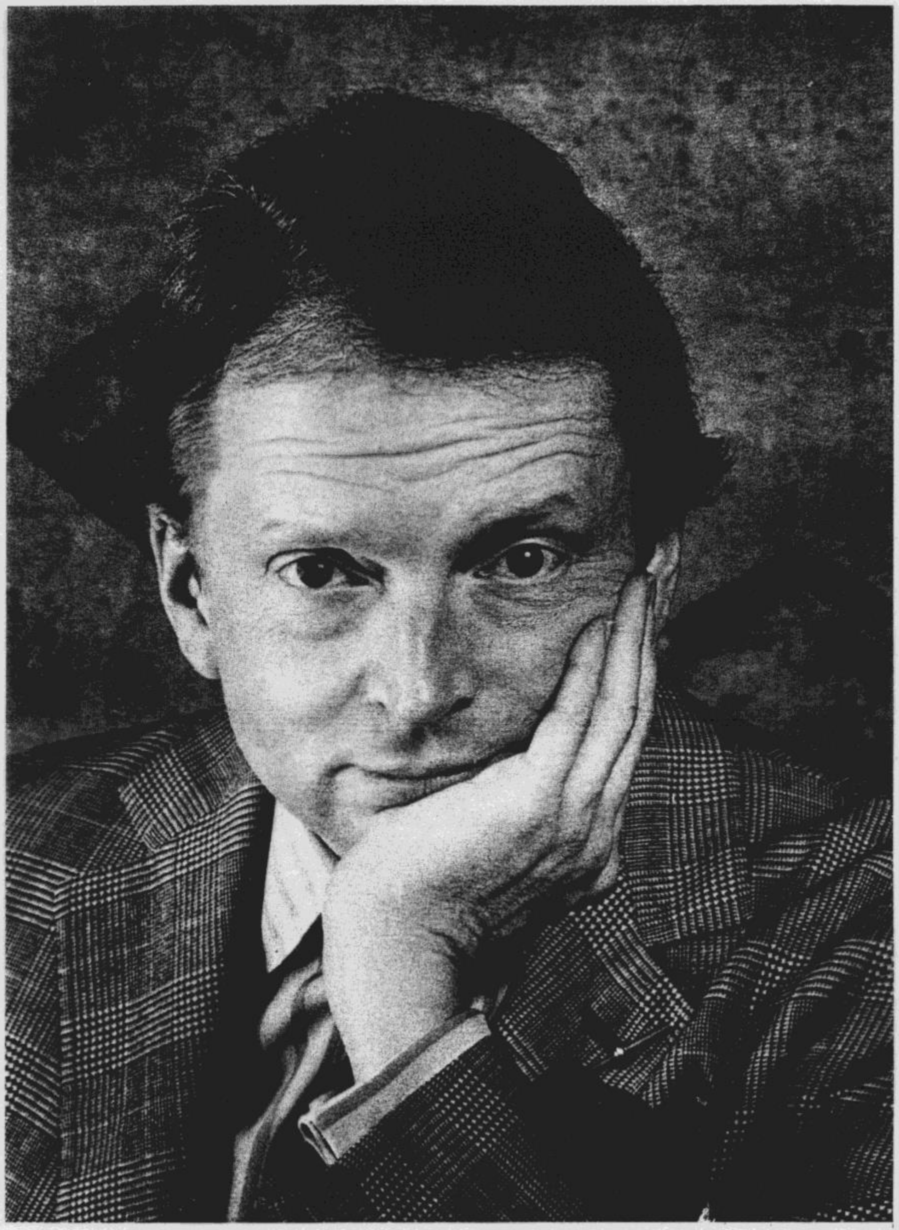
Simple Examples:

Model car
Map of Vermont
Replica of an elephant

Horace Judson: "A model is a rehearsal for reality, a way of making a trial that minimizes the penalties for error."

more extended Judson quote:


The Search for Solutions
Horace Freeland Judson
Introduction by Lewis Thomas, M.D.



PHOTOGRAPH BY SIGRID ESTRADA

HORACE FREELAND JUDSON graduated from the University of Chicago and was Arts and Science Correspondent for Europe for Time magazine in London and Paris. Since 1973 his work has appeared in The New Yorker, Harper's and other magazines. He is author of The Techniques of Reading and Heroin Addiction in England.

1931 - 2011

Quote of the Day

Model-making is a profound and instinctual human response to comprehending the world. Everybody has made paper airplanes, sure. But everybody makes models of many kinds, and all the time. Children make models of the physical and technological world they know, endlessly building and rebuilding with clay, mud and water, blocks, bricks. Children model the social world around them and teach themselves how to get along in it, experimenting and exploring without risk, by their play with dolls. This process is so crucial to children's development that one of the most important tools of child psychiatry, when something seems to be going wrong with the way a small child handles the social world, is the close observation of the model of his life the child offers in playing with a family of dolls. Adolescents will spend countless patient hours constructing models, historical or functional, of wood and paper and string....Throughout adult life, too, people make models, though often in ways that may not be immediately obvious. Scientists and technologists are unusual, here, only because they make models with formal and deliberate seriousness, as professional tools.

Horace Freeland Judson
The Search for Solutions
Page 112
New York: Holt, Rinehart and Winston, 1980

What is a Mathematical Model?

The purpose of our course is to examine how mathematical systems can be used as models: How they are employed to help achieve the goals of science.

Thus, although our orientation is toward applications, you will learn a considerable amount of new mathematics.

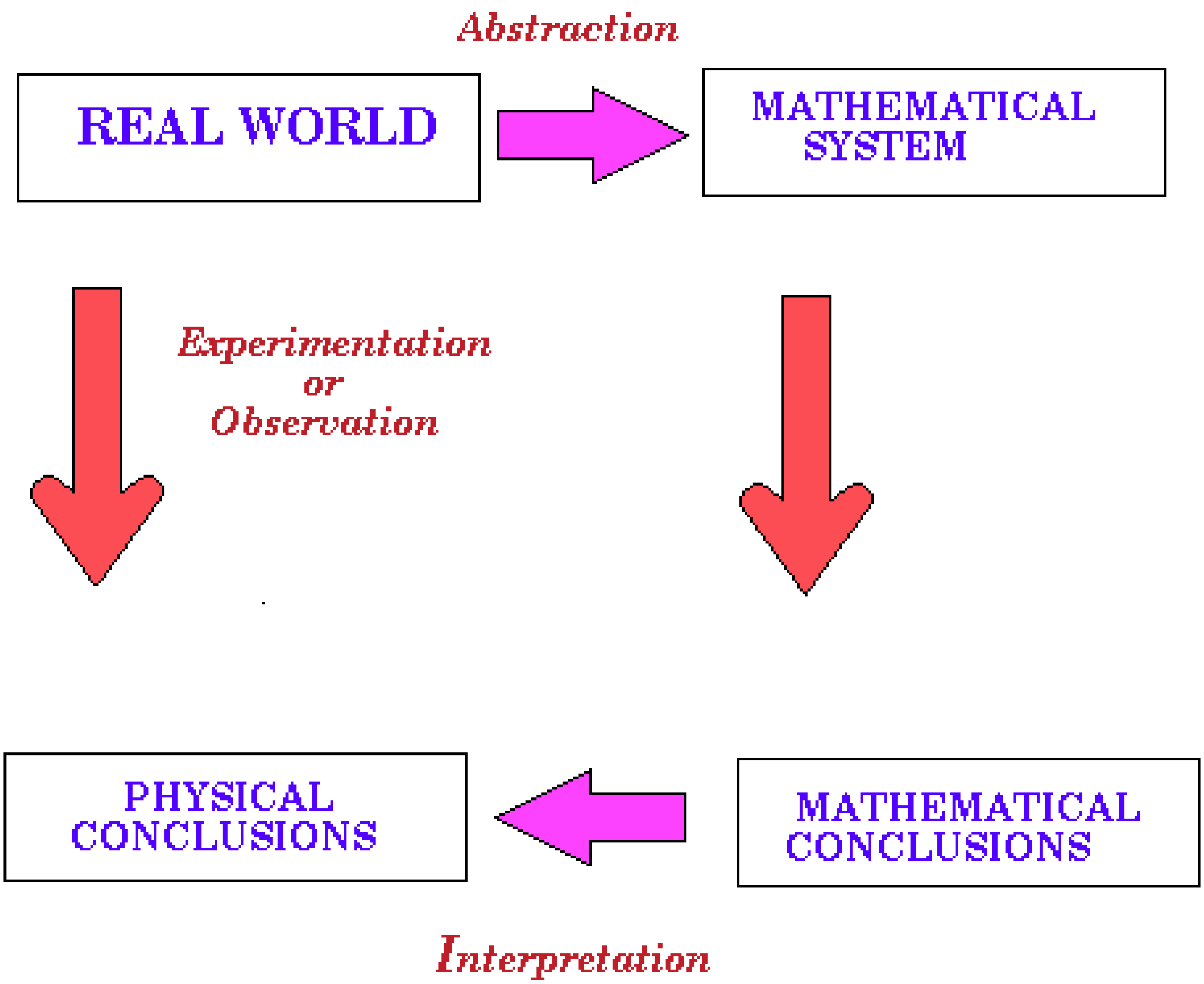
A **mathematical system** consists of a collection of **assertions**, from which **consequences** can be drawn by **logical argument.**

ASSERTIONS: often called Axioms of Postulates of the system. They contain one or more primitive terms which are undefined and have no meaning within the mathematical system.

The axioms of a mathematical system usually consist of statements about the existence of a set of elements, relations among the elements, properties of these relations, and so forth.

Examples: Axioms of Plane Geometry
Vector Spaces
Groups.

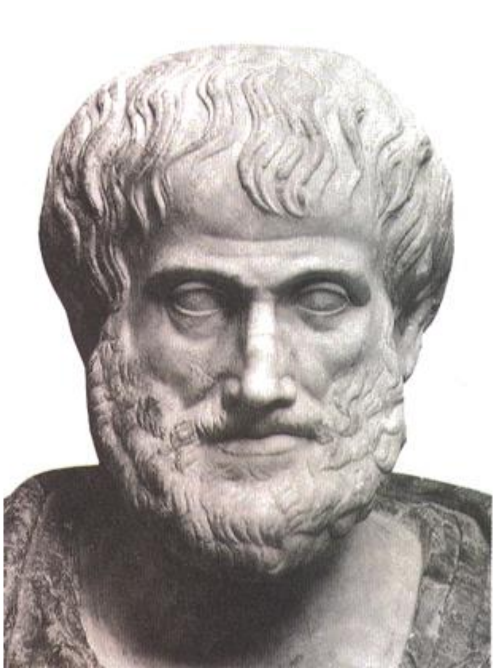
The role that mathematical modeling plays in a science is illustrated by the following diagram:

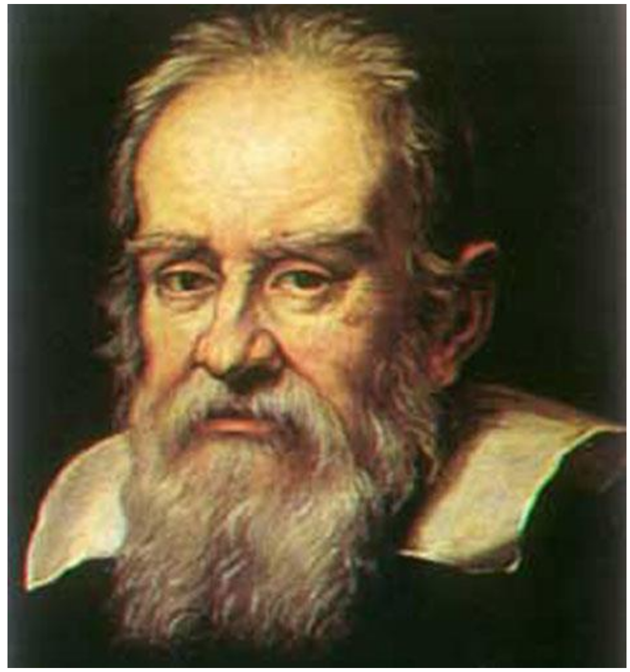


The process:

(1) Abstraction from Real World
(2) Logical Argument to abstract conclusion
(3) Return to real world by process of interpretation.
(4) Comparison of real world with predictions.
(5) Modify step (1) and continue.

The Falling Body Problem



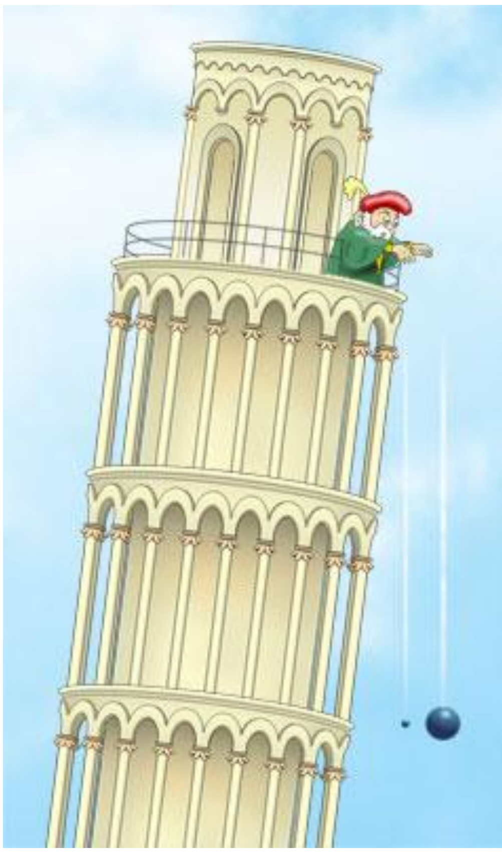


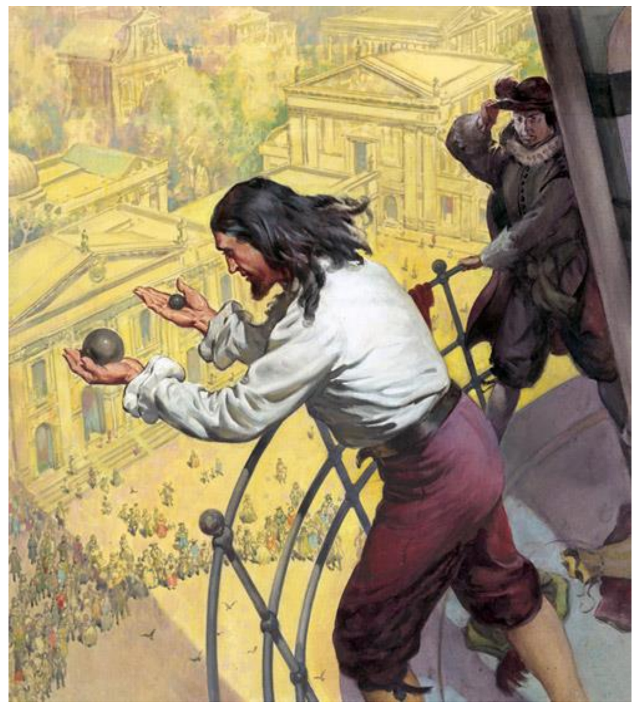
Aristotle vs. Galileo

The motion of falling bodies

Galileo realized, even during his earliest studies (published in his book *On motion*) that the speed of a falling body is *independent* of its weight . He argued as follows: suppose, as Aristotle did, that the manner in which a body falls does depend on its weight (or on some other quality, such as its ``fiery'' or ``earthy'' character), then, for example, a two pound rock should fall faster than a one pound rock. But if we take a two pound rock, split it in half and join the halves by a light string then on the one hand this contraption should fall as fast as a two pound rock, but on the other hand it should fall as fast as a one-pound rock. Since any object should have a definite speed as it falls, this argument shows that the Aristotle's assumption that the speed of falling bodies is determined by their weight is inconsistent; it is simply wrong. Two bodies released from a given height will reach the ground (in general) at different times not because they have different ``earthliness'' and ``fiery'' characteristics, but merely because they are affected by air friction differently. If the experiment is tried in vacuum *any* two objects when released from a given height, will reach the ground simultaneously (this was verified by the Apollo astronauts on the Moon using a feather and a wrench).

This result is peculiar to gravity, other forces do not behave like this at all. For example, if you kick two objects (thus applying a force to them) the heavier one will move more slowly than the lighter one. In contrast, objects being affected by gravity (and starting with the same speed) will have the same speed at all times. This unique property of gravity was one of the motivations for Einstein's general theory of relativity

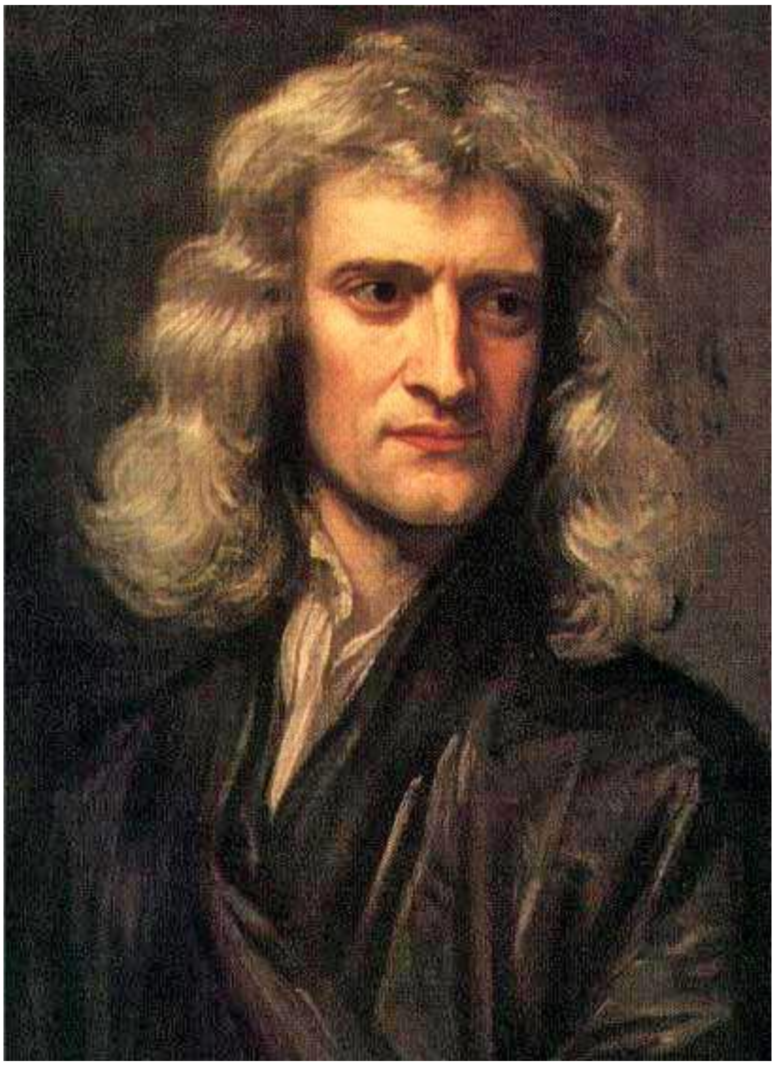




View also: http://www.youtube.com/watch?v=H7de1sTeD6w

Example: Isaac Newton (December 25, 1642 - March 20, 1727)
and the Apple.

See "Newton's Apple and Galileo's Dialogue,"
Stillman Drake, *Scientific American* (August 1980).



Real World: An object drops to the ground from a certain height.

What do we want to know?

Path of Motion?

Speed of Falling Object?

How is the time of fall affected by

- *The weight of the object*
- *Its initial distance above the ground*

Mathematical System

(1) Define variables: What are the important concepts?

(2) Formulate equations: How do the variables interrelate?

Variables:

y The distance of the object above the ground

t time

Note: Both variables are, to some degree at least, observable.

We can measure the initial height.

We can measure elapsed time until object strikes the ground.

Experiments by Gallileo (1564 - 1642)

See Richard Morris, *Time's Arrows: Scientific Attitudes Toward Time.*

Gallileo discovered that time could be used as a tool in the analysis of motion of physical bodies.

Axioms or Assertions

(1) Newton's Second Law of Motion: $\boldsymbol{F = ma}$

(2) Force = force due to gravity

= -32m feet/sec/sec

(3) Acceleration = a = y''

The model is the differential equation

$$y'' = -32.$$

Axioms or Assertions

(1) Newton's Second Law of Motion: $\boldsymbol{F = ma}$

(2) Force = force due to gravity = -32m feet/sec/sec

(3) Acceleration = $a = y''$

The model is the differential equation y" = -32.

Now Mathematical Argument (Solve the Equation)

$$y' = -32t + C$$

$$y = -16\, t^2 + Ct + D$$

Interpretations: C = initial velocity = v_o

D = initial position = y_o

Hence if the object is simply released (C = 0), then

$$y = -16\, t^2 + y_o$$

If the object is simply released ($C = 0$), then

$$y = -16\,t^2 + y_o$$

PREDICTION: How long does it take the apple to fall?

When apple hits ground, $y = 0$ and $t = t_F$

Thus

$$0 = -16t_F^2 + y_0$$

$$16t_F^2 = y_0$$

$$t_F = \frac{\sqrt{y_o}}{4} \text{ seconds}$$

Possible Experimental Verifications

$$tF = \frac{\sqrt{yo}}{4} \text{ seconds}$$

y_O	t_F	Impact Velocity
9 feet	3/4 second	16 mph
100 feet	2.5 seconds	55 mph
1454 feet (Sears Tower)	~9.5 seconds	207 mph
29,028 feet Mt Everest	~ 42 seconds	
93,000,000 miles	48 hours	
3 million lightyears (E.T. 's home)	2416 years	

Where The Model Breaks Down:

(a) Extreme Distance or Size

force of gravity is not constant

Existence of other attracting bodies

Where The Model Breaks Down:

(a) Extreme Distance or Size
force of gravity is not constant
Existence of other attracting bodies

(b) Air Resistance: Crumpled paper vs smooth

More sophisticated models: incorporate these elements into the basic differential equation.

Where The Model Breaks Down:

(a) Extreme Distance or Size
force of gravity is not constant
Existence of other attracting bodies

(b) Air Resistance: Crumpled paper vs smooth

More sophisticated models: incorporate these elements into the basic differential equation.

This still gives a **deterministic model:** the model yields a precise prediction for the time of fall: the entire future behavior of the system is uniquely determined at the outset.

A modern physicist might give you a **probabilistic (stochastic) model** whose prediction would be a probability distribution for arrival times.

Usefulness of a mathematical model can often be judged by the variety of situations to which it can be applied.

Example: Initial velocity nonzero

(a) Robert Adair, *The Physics of Baseball*

(b) Artillery Guns (Ballistic Missiles?)

In our analysis of the model **y'' = -32,** we used the tools of calculus to solve the differential equation for **y** as an explicit function of **t**.

When we refine the model to incorporate other forces or an object of nonconstant mass, we may arrive at a differential equation we do not know how to solve. In such a case, we might try a **simulation** approach.

We'll illustrate this approach with *MATLAB*.

Modeling Projectile Motion

Classic Model: y'' = -g

Using https://www.mathworks.com/help/symbolic/solve-differential-equation-numerically-1.html as template

```
clc
hold off
syms y(t)
g = 32
```

g = 32

```
[V] = odeToVectorField( diff(y,2) == -g  );
M = matlabFunction(V, 'vars', {'t', 'Y'});
Init_Height = 2
```

Init_Height = 2

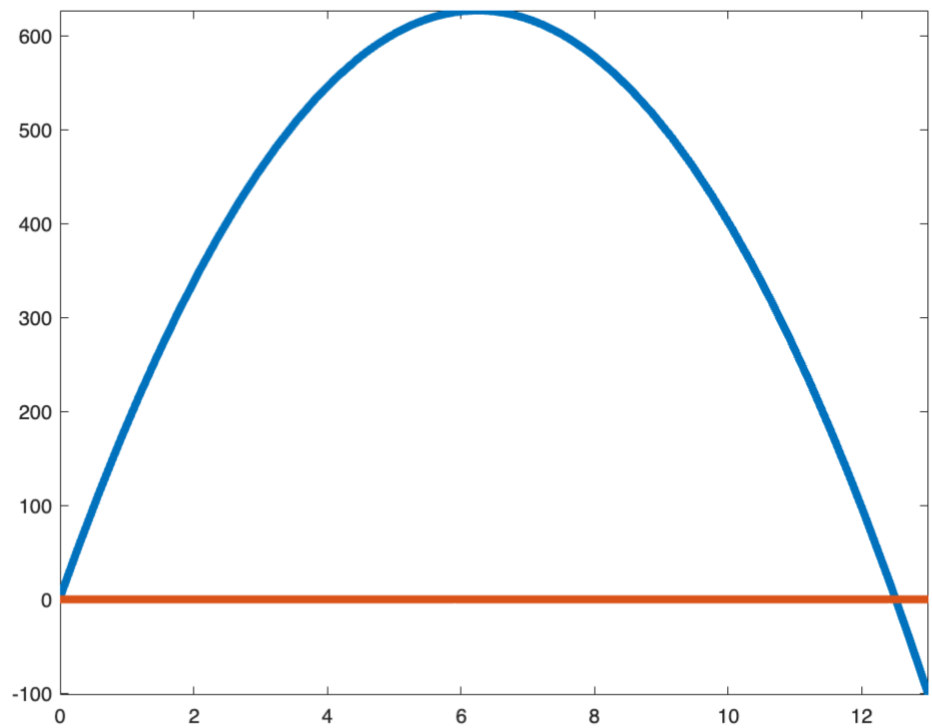
```
Init_Velocity = 200
```

Init_Velocity = 200

```
End_Time = 13
```

End_Time = 13

```
sol = ode45(M,[0 End_Time],[Init_Height Init_Velocity]);
fplot(@(x)deval(sol,x,1), [0, End_Time], 'LineWidth', 4)
hold on
fplot(0, [0,End_Time], 'LineWidth', 4)
hold off
```




```
for c = 1:15
   Time(c) = sol.x(c);
   Height(c) = sol.y(1,c);
   Velocity(c) =sol.y(2,c);
end
 Time = transpose(Time);
 Height = transpose(Height);
 Velocity = transpose(Velocity);
table(Time, Height, Velocity)
```

```
ans = 15×3 table
```

	Time	Height	Velocity
1	0	2	200
2	0.0020	2.4018	199.9357
3	0.0121	4.4091	199.6142
4	0.0623	14.3969	198.0066
5	0.3135	63.1243	189.9685
6	1.5694	276.4757	149.7783
7	2.8694	444.1476	108.1783
8	4.1694	557.7394	66.5783
9	5.4694	617.2513	24.9783
10	6.7694	622.6831	-16.6217
11	8.0694	574.0350	-58.2217
12	9.3694	471.3068	-99.8217
13	10.6694	314.4987	-141.4217
14	11.9694	103.6105	-183.0217
15	13	-102	-216

Classic Model: y'' = -g

Using https://www.mathworks.com/help/symbolic/solve-differential-equation-numerically-1.html as template

```
syms y(t)
g = 32
```

g = 32

```
[V] = odeToVectorField( diff(y,2) == -g  );
M = matlabFunction(V, 'vars', {'t', 'Y'});
Init_Height = 2
```

Init_Height = 2

```
Init_Velocity = 200
```

Init_Velocity = 200

```
End_Time = 13
```

End_Time = 13

```
sol = ode45(M,[0 End_Time],[Init_Height Init_Velocity]);
fplot(@(x)deval(sol,x,1), [0, End_Time], 'LineWidth', 4)
hold on
fplot(0, [0,End_Time], 'LineWidth', 4)
xlabel('Time')
ylabel('Height Above Ground')
ylim([0 650])
```


Height Above Ground
Time
0
100
200
300
400
500
600
0
2
4
6
8
10
12

Retarding Force Proportional To Velocity

$$y'' = -g - p\,y'$$

```
clc
syms y(t)
g = 32
```

g = 32

```
p = .3
```

p = 0.3000

```
[V] = odeToVectorField( diff(y,2) == -g - p * diff(y) )
```

V =

$$\begin{pmatrix} Y_2 \\ -\frac{3\,Y_2}{10} - 32 \end{pmatrix}$$

```
M = matlabFunction(V, 'vars', {'t', 'Y'});
Init_Height = 2
```

Init_Height = 2

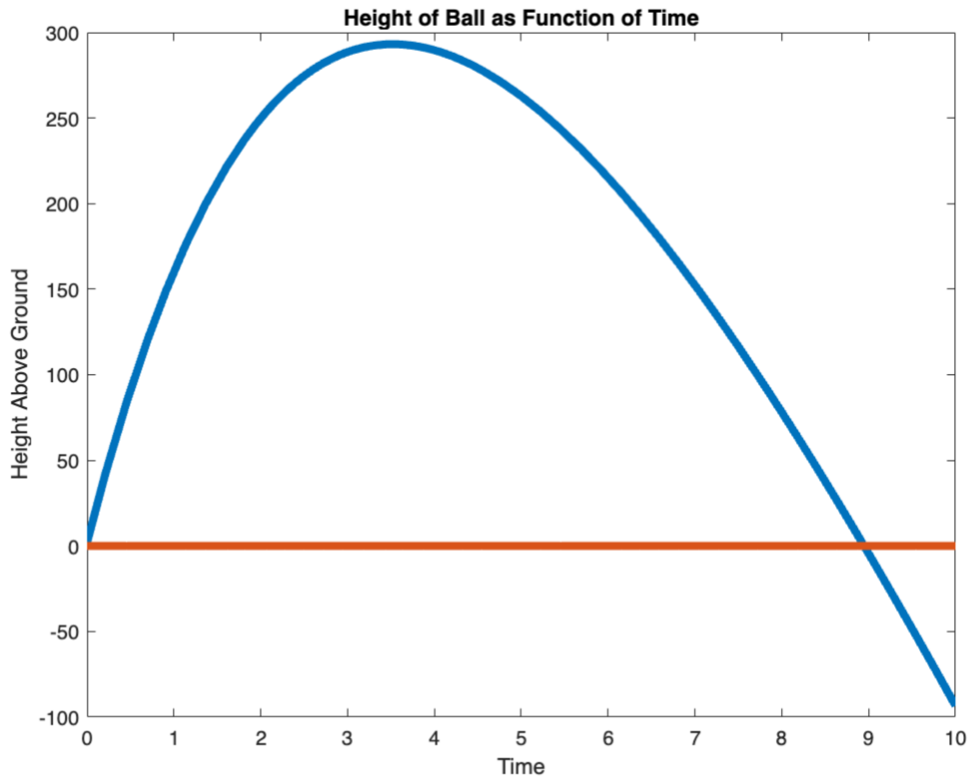
```
Init_Velocity = 200
```

Init_Velocity = 200

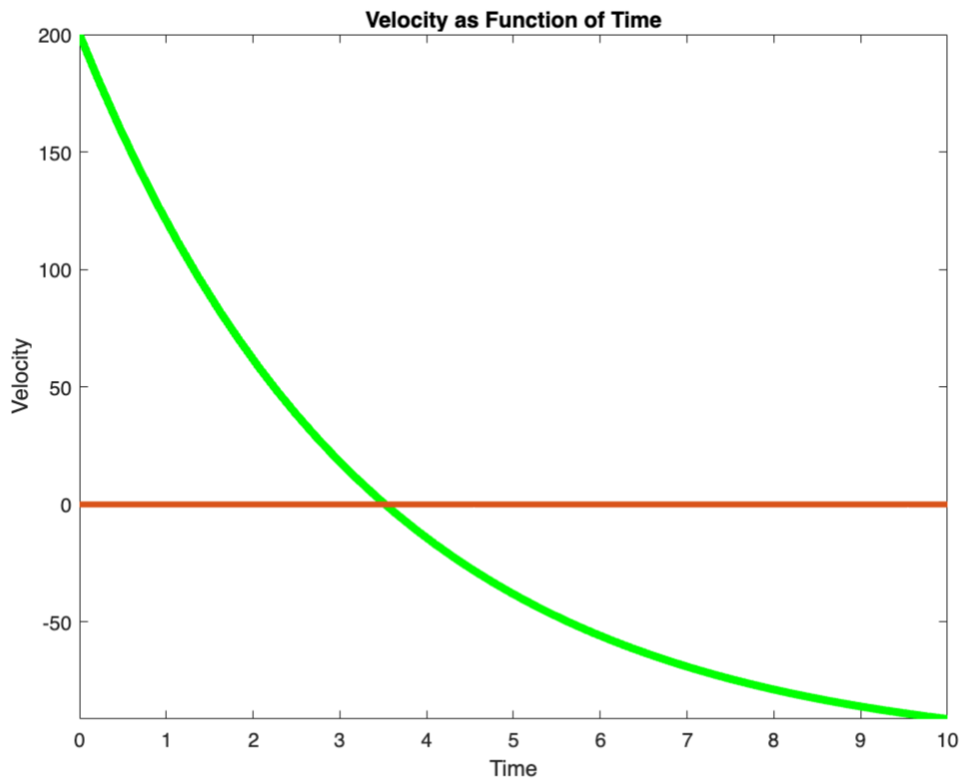
```
End_Time = 10
```

End_Time = 10

```
sol = ode45(M,[0 End_Time],[Init_Height Init_Velocity]);
fplot(@(x)deval(sol,x,1), [0, End_Time], 'LineWidth', 4)
hold on
fplot(0, [0,End_Time], 'LineWidth', 4)
title("Height of Ball as Function of Time")
xlabel("Time")
ylabel("Height Above Ground")
hold off
```




```
fplot(@(x)deval(sol,x,2), [0, End_Time], 'LineWidth', 4, 'Color', 'g')
hold on
fplot(0, [0,End_Time], 'LineWidth', 3)
xlabel("Time")
ylabel("Velocity")
title("Velocity as Function of Time")
```




```
for c = 1:15
   T(c) = sol.x(c);
   H(c) = sol.y(1,c);
   W(c) = sol.y(2,c) ;
end
 Time = transpose(T);
 Height = transpose(H);
 Velocity = transpose(W);
 table(Time, Height, Velocity)
```

ans = 15×3 table

	Time	Height	Velocity
1	0	2	200
2	0.0020	2.4017	199.8152
3	0.0121	4.4047	198.8928
4	0.0623	14.2816	194.3221
5	0.3135	60.3146	172.4741
6	1.3135	194.8084	100.1260
7	2.3135	266.7979	46.5292

	Time	Height	Velocity
8	3.3135	292.4830	6.8236
9	4.3135	283.8650	-22.5910
10	5.3135	249.8345	-44.3818
11	6.3135	196.9781	-60.5249
12	7.3135	130.1751	-72.4840
13	8.3135	53.0402	-81.3435
14	9.3135	-31.7489	-87.9068
15	10	-93.3380	-91.3986