Sample Final Examination Questions *Mathematical Models*
December 2024 MATH 315

1. For which models we have studied is **equilibrium** (or **stability**) an important concept? How is equilibrium defined in each of these cases? In which ones is equilibrium the starting point of the analysis? For which is it the apparent goal? Why do you think mathematical social scientists regard equilibrium –however variously defined – as a critical idea? Be as specific as possible in referring to the particular models studied.

2. Do Parts (a) and (b).

(a) Suppose that the population of the world is now 8 billion and its doubling period is 35 years. If the population is growing at a constant percentage rate, what is that annual rate? What will the population of the world be after 350 years, 700 years, 1050 years? If the surface area of the earth is 1,860,000 billion square feet, how much space would each person get after 1050 years?

(b) The population of New York City would satisfy the logistic law

$$\frac{dp}{dt} = \frac{p}{25} - \frac{p^2}{(25)10^6}$$

where t is measured in years, if we neglected the high emigration and homicide rates.

(i) Modify this equation to take into account the fact that 6,000 people per year move from the city, and 4,000 people per year are murdered.

(ii) Assume that the population of New York City was 8,000,000 in 2010. Find the population for all future time. What happens as $t \to \infty$?

3. Markov Processes:

a. Give a careful definition of each of the following:
   (i) Markov Process
   (ii) Regular Markov Process
   (iii) Absorbing Markov Process

b. Give a clear and complete proof that no Markov process with at least two states can be both regular and absorbing.

c. Give an example of the transition matrix for a Markov process which is neither regular nor absorbing.

4. The game-theoretic analysis of Chapter 22 of *Genesis* discussed four possible outcomes to the story:

A : (Abraham offers Isaac, God is merciful)
B : (Abraham offers Isaac, God is adamant)
C : (Abraham withholds Isaac, God is merciful)
D : (Abraham withholds Isaac, God is adamant)

Suppose that God's preference ordering for these four outcomes is $A > B > D > C$, but that Abraham's faith is rather weak in comparison to his love for his son.

a. Show that a reasonable ordering of the outcomes for Abraham would be $A > C > D > B$.

b. Briefly describe the four strategies that God has in this game.

c. Construct the 2 x 4 payoff matrix consistent with these assignments of preference orderings and determine if either of the players has a dominant strategy. According to the game you have just analyzed, did Abraham and God act as "rational" players in their choice of strategies?

d. Show that no matter how God rank orders the outcomes, one of God's four strategies is always dominant.

6. Describe the verbal assumptions of the classical **PREDATOR-PREY** model.

This model has two critical points, one at a point (x^*,y^*) in the positive first quadrant. Using the Taylor series expansion, we discovered the nature of a trajectory near (x^*,y^*). Briefly describe the nature of such a trajectory.

Using the Taylor series idea, determine the nature of a trajectory near the other critical point (0,0).

7. Discuss Hoffmann's deterministic and probabilistic models of stability in the Galla tribes of Ethiopia. How is the notion of ***stability*** defined in these two models? What is the governing equation of the deterministic model and how is it derived? How did Hoffmann go about deriving the transition matrix for the probabilistic model from raw census data? Briefly summarize Hoffmann's conclusions. What are the strengths and weaknesses of Hoffmann's two approaches?

8. Do parts (a) (b) and (c). They are (essentially) independent of each other.

a. (From Weinstein and Fineberg, *Clinical Decision Analysis*): "physical abuse of children is a serious public health problem. The potential damage caused by allowing a case of child abuse to go undetected is great, but the costs of falsely accusing a parent are also high. As the pediatrician responsible for a school health program, should you institute a screening program of physical examinations to detect abused children? Clearly your response will depend to a large extent on the fraction of children with positive test results who are actually abused.

"The experience of school officials indicates that a careful physical examination will detect 95 percent of battered children (i.e., a false-negative rate of 5 percent), with a false-positive rate of only 10 percent. The best information suggests that 3 percent of school children in an average American city are being abused by their parents."

Suppose a child is examined and found positive, i.e., is diagnosed as having been abused. How likely is it that such a child is, in fact, abused?

b. Appreciative of your background in mathematical modeling, a Las Vegas casino hires you for a J-Term internship. The casino is considering adding a new game: a customer flips a fair coin repeatedly until it lands Tails. If the first Tails occurs on the kth flip, then the customer will receive \$$2^k$. You are asked to determine how much money the customer should be charged in advance to play this game. (If the customer pays \$5 to play ,for example, and flips two Heads and then a Tail, the casino pays \$8 and loses \$3 on the transaction). The casino wants to set the entry fee so that in the long term it will make a profit, but wants it low enough so that potential customers will be willing to play. What's your advice and why?

c. Unable to come up with a satisfactory analytical solution to the problem in (b), you decide to do a simulation of the game to see what the average payoff by the casino might be like. Give a careful and complete description, showing relevant diagrams and equations, of how you would set up such a simulation in MATLAB that would generate 1000 rounds of the game and keep track of the total amount of money paid out by the simulated casino.

10. Exhibit the transition diagram for the Blumstein-Larson recidivism model.

With no assumptions about the equality of P_{R1}, P_{R2} and P_{R3}, find $P(A|I)$, the probability that the person will be arrested at least once more given that he has just been incarcerated.

With $P_A = 1/3$ and $P_I = 1/5$ and $P_{R1} = P_{R2} = P_{R3} = 1/4$, show that the transition matrix takes the form:

	No Recidivism	Crime	Arrest	Prison
No Recidivism	1	0	0	0
Crime	1/2	1/6	1/3	0
Arrest	3/5	1/5	0	1/5
Prison	3/4	1/4	0	0

Suppose a person has just committed a crime. If his behavior is modeled by the transition matrix above, what is the expected number of arrests he will experience. What is the expected number of additional crimes he will commit?

Hint: Assume that $$\begin{pmatrix} 5/6 & -1/3 & 0 \\ -1/5 & 1 & -1/5 \\ -1/4 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 4/3 & 4/9 & 4/45 \\ 1/3 & 10/9 & 10/45 \\ 1/3 & 1/9 & 46/45 \end{pmatrix}$$

11. The human immunodeficiency virus, HIV, leads to acquired immunodeficiency syndrome, AIDS. When antibodies to HIV are detected, the patient is infected and said to be *seropositive* or *HIV positive.* A major problem with AIDS is the variable length of the incubation period from the time the patient is diagnosed as seropositive until the patient exhibits the symptoms of AIDS.

Consider a population in which all of the people are infected with HIV at time t = 0. Let y(t) denote the fraction of the population who have AIDS at time t and x(t) denote the fraction who are seropositive but do not yet have AIDS. Let v(t) be the rate of conversion from infection to AIDS.

a. Show that a reasonable model for this process is

$$dx/dt = -v(t)\,x$$
$$dy/dt = v(t)\,x$$

b. Show that $x(t) + y(t) = 1$ for all t and that $x(0) = 1$ and $y(0) = 0$.

c. If we assume that a patient's immune system to opportunistic diseases, such as cancer, is progressively impaired from the time of infection, show that v(t) would be an increasing function of time.

d. If $v(t) = at$ where $a > 0$ is a constant, determine $x(t)$ and $y(t)$ as explicit functions of t. What portion of the infected people will eventually develop AIDS?

e. Using your solution in (**d**), sketch a graph of dy/dt as a function of t. At what point in time is the growth rate of AIDS patients the largest? What proportion of the population has AIDS at that instant?

12. (a) The Sugar Bush Express provides railroad passenger service between Middlebury and Burlington; Vermont Transit offers bus service between these two communities. A survey of travelers last summer showed that 60% of people who took the train on one trip would take it again on the next trip; 40% would switch to the bus. Of those who had just used the bus, 70% would take the bus again on their next trip while the remainder would switch to the train.

Set up this process as a Markov Chain, identify the states and the steps, display the transition matrix and determine the long term distribution of bus and train travelers.

(b) General Mills is marketing *Star Wars* cereal. In each box of cereal, there is either a Darth Vader or a Luke Skywalker key chain. Suppose there are 10 times as many Lukes as Vaders placed in the boxes. You plan to keep buying boxes of the cereal until you collect both figures. You want to know how many boxes you can expect to buy.

Show that you can answer this question by setting up a Markov Chain whose steps are the successive boxes of cereals and whose states are the sets of figures you have collected so far. Show that the states are Empty Set (neither figure yet collected), {Luke}, {Darth}, and {Luke,Darth}. Determine the probability that a randomly chosen box contains Luke. Use this to find the entries in the transition matrix. Then compute the expected value of the number of boxes you need to purchase.

13. During World War I, F. W. Lanchester developed some mathematical models of combat. In one of these models, Lanchester assumes that there are two combat forces in battle against each other. He assumes that these are "conventional" forces which operate in the open, comparatively speaking, and that every member of a force is within the "kill" range of the enemy. He also assumes that as soon as the conventional force suffers a loss, fire is concentrated on the remaining combatants. Finally, he assumes that each side is reinforced at a constant rate.

(a) Show that Lanchester's assumptions are incorporated in the model

$$dx/dt = -ay + m$$
$$dy/dt = -bx + n$$

where a,b,m and n are positive constants, t represents time, and x and y are the sizes of the two opposing forces.

(b) Graph the stable lines $dx/dt = 0$ and $dy/dt = 0$ in the (x,y)-plane. Determine all the stable points.

(c) Solve the system of equations explicitly in the case that there are no reinforcements;that is, $m = n = 0$.

(d) For the case $a = 1/9$, $b = 1/4$, $m = 8$, $n = 4$, determine the equation of the trajectory for the system by considering the differential equation for dy/dx. What is the shape of the trajectory? What is the long term outcome of the battle?

(e) Comment on the similarities and differences between Lanchester's model and Richardson's model as dynamic models of interaction.

14. (a) Consider a Markov Process with two absorbing states, *A* and *B*, and one transient state. The probability of moving from state *T* to state *A* in a single step is 3/7. The probability of moving from state *T* to state *B* in a single step is 2/7. If the process begins in state *T*, what is the expected number of steps until absorbtion? What is the probability that it will be absorbed into state *A*/

(b) The transition matrix for a certain Markov process is $\mathbf{P} = \frac{1}{6}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$

Find the long term distribution.

(c) Discuss the behavior of a Markov chain whose transition matrix has the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

15. In his book *Nonplussed! Mathematical Proof of Implausible Ideas*, Julian Havil presents the following scenario:

Suppose that there are three members of a tennis club who decide to embark on a private tournament: a new member *M*, his friend *F* (who is a better player) and the club's top player *T*, *M* is encouraged by *F* and by the offer of a prize if *M* wins at least *two games in a row* out of three consecutive games, played alternately against himself and *T*. *M* may choose to begin the series of three games by playing *F* or *T* first.

(a) *M* must choose between the sequence *F T F* or *T F T* as his opponents in the three games. Which should he choose and why?

(b) *M* believes he has probability p of beating *F* and probability q of beating *T* where p < q. What is the probability of *M* winning the prize if he plays *F* first? If he plays *T* first?

(c) What is the expected number of wins for *M* if he plays *F* first? If he plays *T* first?

(d) Suppose the rules are changed so *M* wins the prize if he wins at least two games. Should he choose to play *F* or *T* first?

16. The website http://www.bizarremag.com provides answers to bizarre questions submitted by users. Below is "Doctor Mike's" (no relation) response to "Is it true that there is a certain point when falling from a great height that your body gives up and you die before you hit the ground? "

DOCTOR MIKE: People have free-fallen from heights greater than Mount Everest without dying, but most of them had oxygen and suits designed to protect them from the intense cold of the upper atmosphere. Not to mention parachutes to slow them down on the last stage of the trip.

When falling, a normally clothed body hits its maximum speed of 174 feet per second (about 118 mph, sometimes called terminal velocity) after just twelve seconds, in which time it travels 1583 feet.

This may be 100 feet more than the Petronas Towers in Kuala Lumpur, the tallest building in the world, but skydivers do this kind of fall all the time. It's possible that an unprotected person would die from exposure and/or lack of oxygen on the way down from an enormous fall, but what's more likely is that someone would black out, or at least go into shock, as they are confronted by their impending and unavoidable death.

What "Doctor Mike" appears to have in mind is a modification of the classical model of free fall which includes a term that retards acceleration in a manner proportional to the square of velocity.

(a) Show that a reasonable form for such a model is $dv/dt = 32 - kv^2$ where v is velocity in feet per second and t is time.

(b) Use the data given in Mike's answer to find k.

(c) Solve the differential equation to find v as an explicit function of t. What is the value of v when $t = 12$?

(d) Use your solution in (c) to find the distance s fallen after t seconds as an explicit function of t. How far would you fall in 12 seconds?

17. (a)What does it mean for a player to have a *dominant* strategy in a general two-person *m x n* game?

The matrix below shows the payoffs to Rose in a zerosum 2 x 2 game between row player Rose and column player Colin:

	C1	C2
R1	5	3
R2	-4	-7

(b) Which of the two players, if either, has a dominant strategy? What are the dominant strategies?

(c) How would you advise Rose and Colin to play?

(d) Suppose the entries in the payoff matrix are invisible to you, Rose and Colin. You suggest they follow Anatol Rapoport's dynamic strategy and they agree. Show that their subsequent behavior would be modeled by the system of differential equations

$$\frac{dx}{dt} = 32y + 7$$

$$\frac{dy}{dt} = -32x - 3$$

.

(e) What do *x* and *y* represent in this system? What is the long term outcome? How does the long term outcome of the system of differential equations compare with advice you gave in part(c)?

(f) Show how you would set up this model in *STELLA*, including a display of the diagram and a listing of the relevant *STELLA* equations. Note that *x* and *y* must remain between 0 and 1 to reflect the reality of playing the game; why? Show how you can easily get *STELLA* to insure that $x \geq 0$ and $y \geq 0$. How can you keep *x* and *y* from exceeding 1?

18. Consider the two-person nonzero sum game *Tennis* whose payoff matrix is

	C1	C2
R1	(90,10)	(20,80)
R2	(30,70)	(60,40)

(a) What does it mean for a nozero sum game to have a *Nash equilibrium?*

(b) Is there a Nash equilibrium in pure strategies for this particular game? Explain.

(c) Find the mixed-strategy Nash equilibrium for this game.

(d) Outline Nash's proof that a Nash equilibrium exists for any nonzero sum game among a finite collection of players, each of whom has a finite number of pure strategies.

19. The current (May 29, 1997) issue of *The New York Review of Books* contains an article "Reaching the Limit" by well known environmentalist Bill McKibben. The initial part of the article appears below.

(a) McKibben claims "If we stalled at current fertility rates, the population would reach the absurd size of 700 billion by the year 2150." Using the estimate of 5.7 billion as the current world population and McKibben's assumption of constant percentage growth rate, determine what the current growth rate is.

(b) What model of population growth is consistent with the predictions made in the first paragraph of McKibben's article that world population "essentially stabilize" or "top out" ?

(c) What model of population growth do you think McKibben is using to support the claim in the last sentence of the second paragraph?

13. Do Parts (a) and (b).

(a) Suppose that the population of the world is now 6 billion and its doubling period is 35 years. If the population is growing at a constant percentage rate, what is that annual rate? What will the population of the world be after 350 years, 700 years, 1050 years? If the surface area of the earth is 1,860,000 billion square feet, how much space would each person get after 1050 years?

(b) The population of New York City would satisfy the logistic law

$$\frac{dp}{dt} = \frac{p}{25} - \frac{p^2}{(25)10^6}$$

where t is measured in years, if we neglected the high emigration and homicide rates.

(a) Modify this equation to take into account the fact that 6,000 people per year move from the city, and 4,000 people per year are murdered.

(b) Assume that the population of New York City was 8,000,000 in 2000. Find the population for all future time. What happens as $t \rightarrow \infty$?

2. <u>Hospital Simulation</u>

a. Porter Medical Center in Middlebury is planning an expansion of its facilities which will add 50 new patient beds. The 2005 medical records show that 70% of all hospitalized patients at the center had surgery. About 2 percent of the surgical patients had thyroid surgery and spent an average of 5 days in the hospital. Use the Schmitz-Kwak technique to estimate the additional number of thyroid operations that will be done at Porter in 2009 after it completes the proposed expansion.

b. Suppoose that the relative frequency *y* of operations taking *t* hours to complete is given by a continuous function *f(t)*. Show that a reasonable measure of the epxected or average length of an operation is given by the integral $\int_{o}^{\infty} t\ f(t)\ dt$.

c. For Schmitz and Kwak the functon *f(t)* was □□□e□□□t The Middlebury modelers have discvovered that their relative frequencies are given by the functiuon $y = a\,(1 + b\,t^2)^{-3/2}$ where *a* and *b* are positive constants. Compute the average length of an operation at the Porter Medical Center.

d. Discuss several ways in which the Schmitz and Kwak model could have been made more realistic.