MATH 315 Sample Mid-Term Examination Questions

1. An aggressive investor claims that her fortune increases at a rate proportional to the square of her current wealth.

(a) Discuss why the differential equation $\mathbf{dW/dt = a\, W^2}$ (where $\mathbf{a}$ is a positive constant) is an appropriate mathematical model for the investor's claim.

(b) Without solving the differential equation, discuss the nature of the graph of Wealth vs Time. In particular, show the curve is concave up.

(c) If the investor has a wealth of $\mathbf{W}_o$ at time $\mathbf{t = 0}$, solve the differential equation for $\mathbf{W}$ in terms of $\mathbf{a}$, $\mathbf{t}$ and $\mathbf{W_o}$.

(d) Suppose our investor had \$1 million a year ago, and has \$1.5 million dollars today. How much will she be worth in 6 months? a year?

(e) Based on your analysis in part (d), discuss the validity of the investor's claim.

2. Consider a Richardson arms race with

$$dx/dt = 5y - 7x - 6$$
$$dy/dt = 4x - 2y - 12$$

(a) What verbal assumptions are reflected in this model?

(b) Find the equations of the stable lines L and L', sketch them and determine the point of stability.

(c) Find the outcome of such an arms race if

(i) initial level is (5,5)

(ii) initial level is (13,6)

3. During World War I, F. W. Lanchester developed some mathematical models of combat. In one of these models, Lanchester assumes that there are two combat forces in battle against each other. He assumes that these are "conventional" forces which operate in the open, comparatively speaking, and that every member of a force is within the "kill" range of the enemy. He also assumes that as soon as the conventional force suffers a loss, fire is concentrated on the remaining combatants. Finally, he assumes that each side is reinforced at a constant rate.

(a) Show that Lanchester's assumptions are incorporated in the model

$$dx/dt = -ay + m$$
$$dy/dt = -bx + n$$

where a,b,m and n are positive constants, t represents time, and x and y are the sizes of the two opposing forces.

(b) Graph the stable lines $dx/dt = 0$ and $dy/dt = 0$ in the (x,y)-plane. Determine all the stable points.

(c) Solve the system of equations explicitly in the case that there are no reinforcements;that is, $m = n = 0$.

(d) For the case $a = 1/9$, $b = 1/4$, $m = 8$, $n = 4$, determine the equation of the trajectory for the system by considering the differential equation for dy/dx. What is the shape of the trajectory? What is the long term outcome of the battle?

4. A more realistic model of competitive species is:

$$dx/dt = ax - cx^2 - bxy, \quad dy/dt = my - py^2 - nxy$$

where x and y represent the populations of the two species at time t, and a,b,m,n,c and p are positive constants.

(a) What verbal assumptions does this model reflect? Why are they more realistic than those of the model discussed in the text?

(b) Graph the stable lines $dx/dt = 0$ and $dy/dt = 0$ in the (x,y)-plane. Discuss why there is at most one critical point (x^*,y^*) with both x^* and y^* positive.

(c) The values of the parameters are chosen so that

$$dx/dt = 40x - x^2 - 2xy, \quad dy/dt = 72y - 3y^2 - 2xy$$

Find (x^*,y^*) explicitly.

(d) For the system in (c), discuss the long term behavior if

(i) the initial population is of the form $(0, y_o)$

(ii) the initial population is near (x^*,y^*)

(e) For the system in (c), use the Taylor series approximation to find the linear system whose qualitative behavior near (x^*,y^*) is the same as that of the original system. Do NOT attempt to solve this linear system of differential equations.

5. Mathematical modelers are occasionally asked to testify in legal proceedings. A motorist driving along Weybridge Street made an emergency stop, brakes locked and wheels sliding. The length of skid marks on the road was 19. 2 feet. The speed limit on this section of Weybridge Street is 30 miles per hour (44 feet per second).

The police officer investigating the case made the reasonable assumption that the maximum deceleration of the car would not exceed the acceleration of a freely falling body (32 feet/sec/sec), did some calculations, and arrested the motorist for speeding. Assuming that the only force acting on the car was a constant deceleration, develop a mathematical model for the motion of the car beginning with the instant the brakes were applied.

Do the conclusions of your model support the police officer or do they support the motorist who claims not to have exceeded the speed limit? Can you determine the true speed of the motorist?

6. If P is the population at time t, then the *exponential growth* model is the differential equation $dP/dt = aP$ where a is a constant.

(a) Find the solution of this differential equation.

(b) The *logistic growth* model is a generalization of exponential growth. What does the underlying differential equation for this model look like?

(c) Consider a model of population growth of the form $dP/dt = P(P-a)(P-b)$ where a and b are positive constants with $b > a$. In what ways is this model a generalization of logistic growth. Discuss what happens in the long run under this model if the initial population is

(i) less than a

(ii) between a and b

(iii) greater than b

(d) Determine P explicitly if $dP/dt = P(P-10)(P-20)$ and the initial population $P(0)$ is 15.

7. In a provocative paper (""Coercion and Revolution: Variations on a Predator-Prey Model,' *Mathematical and Computer Modelling*, Volume 12 (1989), pp. 547 - 559), political scientists George Tsebelis and John Sprague discuss mathematical models exploring the interrelationships among revolution, coercion, relative deprivation, and foreign aid. With R representing the level of revolutionary activity and C representing the level of state coercion, one of their models has the form

$$dR/dt = - a R + g C + b$$

$$dC/dt = + h R - k C + c$$

where a,g,b,h,k and c are constants. They assume that the parameters a,b,k and c are positive while the parameters g and h are of opposite sign.

(a) What verbal assumptions are reflected in this model? Do they seem reasonable to you? Why?

(b) Analyze this model using the techniques we have developed under the assumption that g is negative and h is positive. What is the long-term qualitative behavior of this system? Include relevant graphs in a (R,C)-plane.

(c) How is your answer in (a) affected if g is positive and h is negative?

(d) Are the authors justified in describing this model as a "variation" of the predator-prey model?