Solving $y'(t) = ay(t) + b,$ with $y(0) = c$

Method 1: Separate Variables

Rewrite differential equation as $\frac{y\prime(t)}{ay(t)+b} = 1$ and integrate both sides with respect to *t* to obtain
$\frac{1}{a}\ln(a\, y(t) + b) = t + C$
so $\ln(\, a\, y(t) + b) = at + aC$ for some constant C. Applying the exponential function to each side of the equation yields

$a\, y(t) + b = \, e^{at+aC} = K\, e^{at}$ and $y(t) = \frac{Ke^{at}-b}{a}$. Then use $y(0) = c$ to evaluate *K:*

$c = \frac{K-b}{a}$ so $K = ac + b$. Thus $y(t) = \frac{(ac+b)e^{at}-b}{a}$

Method 2: Change of Variable:

Let $x(t) = ay\,(t) + b$. Then $x'(t) = ay'(t)$ and $y'(t) = \frac{1}{a}x'(t)$. The differential equation becomes $\frac{1}{a}x'(t) = x(t)$ or $x' = ax$, the classic exponential growth function whose solution is $x = C\, e^{at}$ for some constant *C* with $x(0) = C\, e^{a0} = C.$ so $x(t) = x(0)e^{at}$ Now $x(0) = ay(0) + b = ac + b.$
Thus $ay(t) + b = (ac + b)e^{at}$ as we obtained with Method 1.

Method 3: Treat equation as first order linear differential equation and use Integrating Factor.

Rewrite the differential equation as $y' - ay = b$. The integrating factor is e^{-at}. Multiply the equation by the integrating factor:

$e^{-at}\, y'^{(t)} - a\, e^{-at}y(t) = \, b\, e^{-at}$ or $(e^{-at}y(t))' = \, b\, e^{-at}$. Integrate both sides: $e^{-at}y(t) = -\frac{b}{a}e^{-at} + C.$

Finally, multiply by e^{at} to get $y(t) = C\, e^{at} - \frac{b}{a} = \frac{aC\, e^{at}-b}{a}$

Use initial condition $y(0) = c$ to determine C as above.